HIGHER SPIN MAPPING CLASS GROUPS AND STRATA OF ABELIAN DIFFERENTIALS OVER TEICHMÜLLER SPACE

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Abstract. For $g \geq 5$, we give a complete classification of the connected components of strata of abelian differentials over Teichmüller space, establishing an analogue of a theorem of Kontsevich and Zorich in the setting of marked translation surfaces. Building off of work of the first author [Cal19], we find that the non-hyperelliptic components are classified by an invariant known as an r–spin structure. This is accomplished by computing a certain monodromy group valued in the mapping class group. To do this, we determine explicit finite generating sets for all r–spin stabilizer subgroups of the mapping class group, completing a project begun by the second author in [Sal19]. Some corollaries in flat geometry and toric geometry are obtained from these results.

1. Introduction

The moduli space $\Omega\mathcal{M}_g$ of abelian differentials is a vector bundle over the usual moduli space $\mathcal{M}_g$ of closed genus g Riemann surfaces, whose fiber above a given $X \in \mathcal{M}_g$ is the space $\Omega(X)$ of abelian differentials (holomorphic 1–forms) on X. Similarly, the space $\Omega\mathcal{T}_g$ of marked abelian differentials is a vector bundle over the Teichmüller space $\mathcal{T}_g$ of *marked* Riemann surfaces of fixed genus (recall that a *marking* of $X \in \mathcal{M}_g$ is an isotopy class of map from a (topological) reference surface Σ_g to X).

Both $\Omega\mathcal{M}_g$ and $\Omega\mathcal{T}_g$ are naturally partitioned into subspaces called *strata* by the number and order of the zeros of a differential appearing in the stratum. For a partition $\underline{\kappa} = (k_1, \ldots, k_n)$ of $2g-2$, define

$$\Omega\mathcal{M}(\underline{\kappa}) := \{(X, \omega) \in \Omega\mathcal{M}_g : \omega \in \Omega(X) \text{ with zeros of order } k_1, \ldots, k_n\}.$$

Define $\Omega\mathcal{T}(\underline{\kappa})$ similarly; then $\Omega\mathcal{M}(\underline{\kappa})$ is the quotient of $\Omega\mathcal{T}(\underline{\kappa})$ by the mapping class group $\operatorname{Mod}(\Sigma_g)$. Each stratum $\Omega\mathcal{M}(\underline{\kappa})$ is an orbifold, and the mapping class group action demonstrates $\Omega\mathcal{T}(\underline{\kappa})$ as an orbifold covering space of $\Omega\mathcal{M}(\underline{\kappa})$.

While strata are fundamental objects in the study of Riemann surfaces, their global structure is poorly understood (outside of certain special cases [LM14]). Kontsevich and Zorich famously proved that there are only ever at most three connected components of $\Omega\mathcal{M}(\underline{\kappa})$, depending on hyperellipticity and the "Arf invariant" of an associated spin structure (see Theorem 3.5 and Definition 2.12).

In [Cal19], the first author gives a partial classification of the non-hyperelliptic connected components of $\Omega\mathcal{T}(\underline{\kappa})$ in terms of invariants known as "r–spin structures" (c.f. Definition 2.1). Our first main theorem finishes that classification, settling Conjecture 1.3 of [Cal19] for all $g \geq 5$.

Theorem A (Classification of strata)**.** *Let $g \geq 5$ and $\underline{\kappa} = (k_1, \ldots, k_n)$ be a partition of $2g-2$. Set $r = \gcd(\underline{\kappa})$. Then there are exactly r^{2g} non–hyperelliptic components of $\Omega\mathcal{T}(\underline{\kappa})$, corresponding to the r–spin structures on Σ_g.*

Moreover, when $\gcd(\underline{\kappa})$ is even, exactly $(r/2)^{2g}\left(2^{g-1}(2^g+1)\right)$ of these components have even Arf invariant and $(r/2)^{2g}\left(2^{g-1}(2^g-1)\right)$ have odd Arf invariant.

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This result should be contrasted with the classification of hyperelliptic components appearing in [Cal19, Corollary 2.6]. In the hyperelliptic case, there are infinitely many connected components for $g \geq 3$, in bijection with hyperelliptic involutions of the surface.

We emphasize that Theorem A together with [Cal19, Corollary 2.6] yields a complete classification of the connected components of $\Omega\mathcal{T}(\underline{\kappa})$ for all $g \geq 5$.

Using the correspondence of Theorem A, we give a complete characterization of which curves can be realized as embedded Euclidean cylinders on some surface in a given connected component of a stratum (see Section 3.1 for a discussion of cylinders and other basic notions in flat geometry).

Corollary 1.1. *Suppose that $g \geq 5$ and $\underline{\kappa}$ is a partition of $2g-2$ with $\gcd(\underline{\kappa}) = r$. Let $\widetilde{\mathcal{H}}$ be a component of $\Omega\mathcal{T}(\underline{\kappa})$ and ϕ the corresponding r–spin structure, and let $c \subset \Sigma_g$ be a simple closed curve.*

- *If $\widetilde{\mathcal{H}}$ is hyperelliptic with corresponding involution ι, then c is realized as the core curve of a cylinder on some marked abelian differential in $\widetilde{\mathcal{H}}$ if and only if it is nonseparating and $\iota(c) = c$.*
- *If $\widetilde{\mathcal{H}}$ is non-hyperelliptic, then c is realized as the core curve of a cylinder on some marked abelian differential in $\widetilde{\mathcal{H}}$ if and only if it is nonseparating and $\phi(c) = 0$.*

The proof of Theorem A follows by analyzing which mapping classes can be realized as flat deformations living in a (connected component of a) stratum. In particular, we show that the "geometric monodromy group," a kind of homotopy analogue for the monodromy of the Gauss–Manin connection over $\Omega\mathcal{M}(\underline{\kappa})$, is equal to the stabilizer of an r–spin structure under the natural $\mathrm{Mod}(\Sigma_g)$ action (see Definitions 3.6 and 2.5).

Towards this goal, our second main theorem provides explicit finite generating sets for the stabilizer of any r–spin structure. In [Sal19, Theorem 9.5], the second author obtained partial results in this direction, but the results there only applied in the setting of $r < g-1$, and were only approximate in the case of r even.

To state our results, we recall that the set of r–spin structures on Σ_g is empty unless r divides $2g-2$ (see Remark 2.2). For any r–spin structure ϕ on a surface of genus g, define a *lift* of ϕ to be any $(2g-2)$–spin structure $\widetilde{\phi}$ such that

$$\widetilde{\phi}(c) \equiv \phi(c) \pmod{r}$$

for every oriented simple closed curve c. Let $\mathrm{Mod}_g[\phi]$ denote the stabilizer of ϕ under the natural $\mathrm{Mod}(\Sigma_g)$ action (see Definition 2.5).

Since $\mathrm{Mod}(\Sigma_g)$ acts transitively on the set of r–spin structures with the same Arf invariant (Lemma 2.15), it suffices to give generators for the stabilizer of a single r–spin structure with given Arf invariant.

Theorem B (Generating $\mathrm{Mod}_g[\phi]$)**.**

(1) Let $g \geq 5$ be given. Then there is a $(2g-2)$–spin structure ϕ with

$$\mathrm{Arf}(\phi) = \begin{cases} 1 & g \equiv 0,3 \pmod{4} \\ 0 & g \equiv 1,2 \pmod{4} \end{cases}$$

such that $\mathrm{Mod}_g[\phi]$ *is generated by the finite collection of Dehn twists* $a_0, a_1, \ldots, a_{2g-1}$ *shown in Figure 1.*

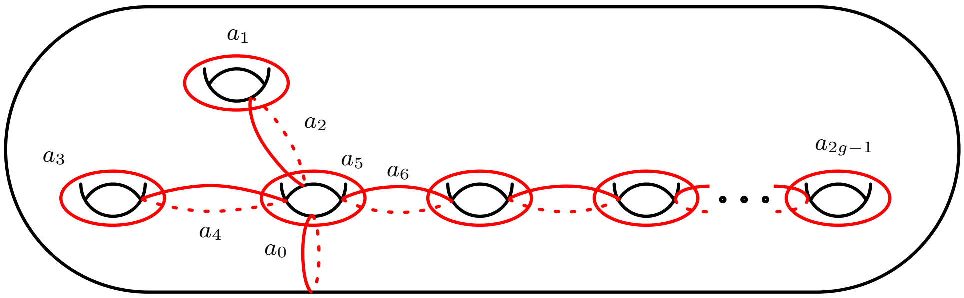

FIGURE 1. Generators for $\mathrm{Mod}_g[\phi]$, Case 1

(2) *Let* $g \geq 5$ *be given. Then there is a* $(2g-2)$*–spin structure* ϕ *with*

$$\mathrm{Arf}(\phi) = \begin{cases} 1 & g \equiv 1, 2 \pmod 4 \\ 0 & g \equiv 0, 3 \pmod 4 \end{cases}$$

such that $\mathrm{Mod}_g[\phi]$ *is generated by the finite collection of Dehn twists* $a_0, a_1, \ldots, a_{2g-1}$ *shown in Figure 2.*

(3) *Let* $g \geq 3$*, let* r *be a proper divisor of* $2g-2$*, and let* ϕ *be an* r*–spin structure. Let* $\widetilde{\phi}$ *be an arbitrary lift of* ϕ *to a* $(2g-2)$*–spin structure, and let* $\{c_i\}$ *be any collection of simple closed curves such that the set of values* $\{\widetilde{\phi}(c_i)\}$ *generates the subgroup* $r\mathbb{Z}/(2g-2)\mathbb{Z}$*. Then*

$$\mathrm{Mod}_g[\phi] = \langle \mathrm{Mod}_g[\widetilde{\phi}], \{T_{c_i}\} \rangle.$$

In particular, $\mathrm{Mod}_g[\phi]$ *is generated by a finite collection of Dehn twists for all* $g \geq 5$*: the twists about the curves* $\{c_i\}$ *in combination with the finite generating set for* $\mathrm{Mod}_g[\widetilde{\phi}]$ *given by whichever of Theorem B.1 or B.2 is applicable to* $\widetilde{\phi}$*.*

In the course of proving Theorem B, we establish in Proposition 6.1 that the group $\mathcal{T}_\phi$ of "admissible twists" (c.f. Definition 2.6) generated by the Dehn twists in all nonseparating curves c with $\phi(c) = 0$

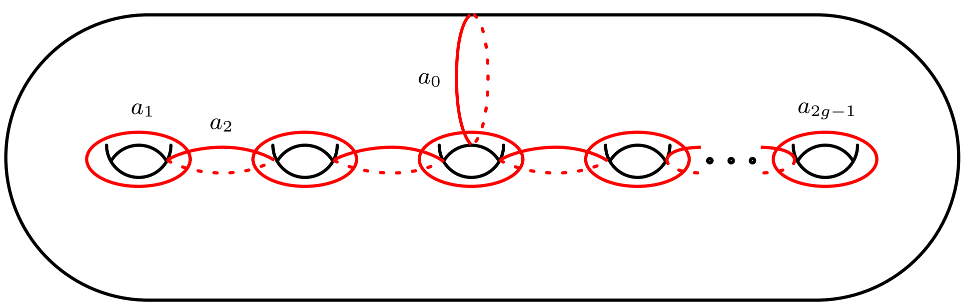

FIGURE 2. Generators for $\mathrm{Mod}_g[\phi]$, Case 2

is equal to the spin stabilizer subgroup $\mathrm{Mod}_g[\phi]$. Together with the main theorem of [Sal19], this is enough to settle a conjecture of the second author. We refer the interested reader to [Sal19] for the relevant definitions.

Corollary 1.2 (c.f. Conjecture 1.4 of [Sal19])**.** *Suppose that $\mathcal{L}$ is an ample line bundle on a smooth toric surface Y for which the generic fiber $\Sigma_{g(\mathcal{L})}$ has genus at least 5 and is not hyperelliptic.*

Let $|\mathcal{L}|$ denote the complete linear system of $\mathcal{L}$ and $\mathcal{M}(\mathcal{L}) \subset |\mathcal{L}|$ the complement of the discriminant locus; then $\mathcal{M}(\mathcal{L})$ supports a tautological family of Riemann surfaces. Let $\pi : \mathcal{E}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$ be the corresponding $\Sigma_{g(\mathcal{L})}$ bundle, and let

$$\Gamma_{\mathcal{L}} \leqslant \mathrm{Mod}(\Sigma_{g(\mathcal{L})})$$

denote the image of the monodromy representation of π. Then

$$\Gamma_{\mathcal{L}} = \mathrm{Mod}_g[\phi],$$

where ϕ is the r–spin structure induced by the adjoint line bundle $\mathcal{L} \otimes K_Y$.

Proof. By [Sal19, Theorem A], $\mathcal{T}_\phi \leqslant \Gamma_{\mathcal{L}} \leqslant \mathrm{Mod}_g[\phi]$. By Proposition 6.1, $\mathcal{T}_\phi = \mathrm{Mod}_g[\phi]$. □

As a final corollary, we recover a recent theorem of Gutiérrez–Romo [GR18, Corollary 1.2] using topological methods. The result of Gutiérrez–Romo concerns the *homological* monodromy of a stratum. Let $\mathcal{H}$ be a component of $\Omega\mathcal{M}(\underline{\kappa})$. There is a vector bundle $H_1\mathcal{H}$ over $\mathcal{H}$ where the fiber over the Abelian differential (X, ω) is the space $H_1(X, \mathbb{R})$. The (orbifold) fundamental group of $\mathcal{H}$ admits a monodromy action on $H_1\mathcal{H}$ as a subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$; this was computed (via the "Rauzy–Veech group" of $\mathcal{H}$) by Gutiérrez–Romo.

Corollary 1.3 (c.f. Corollary 1.2 of [GR18])**.** *Suppose that $\underline{\kappa} = (k_1, \dots, k_n)$ is a partition of $2g - 2$ such that $g \geq 5$, and set $r = \gcd(k_1, \dots, k_n)$. Let $\mathcal{H}$ be a connected component of $\Omega\mathcal{M}(\underline{\kappa})$.*

(1) If r is odd, then the monodromy group of $H_1\mathcal{H}$ is the entire symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$.
(2) If r is even, then the monodromy group of $H_1\mathcal{H}$ is the stabilizer in $\mathrm{Sp}(2g, \mathbb{Z})$ of a quadratic form q associated to the spin structure on the chosen basepoint (see Section 2.3).

The proof of Corollary 1.3 follows essentially immediately from Theorem A; see the end of Section 6.1 for details.

Relation to previous work. The present paper should be viewed as a joint sequel to the works [Cal19] and [Sal19]. So as to avoid a large amount of redundancy, we have aimed to give an exposition that is self–contained but does not dwell on background. The reader looking for a more thorough discussion of flat geometry is referred to [Cal19], and the reader looking for more on r–spin structures is referred to [Sal19]. We have also omitted the proofs of many statements that are essentially contained in our previous work. In some cases we require slight modifications of our results that cannot be cited directly; in this case, we have attempted to indicate the necessary modifications without repeating the arguments in their entirety.

For the most part, the technology of [Cal19] does not need to be improved, and much of the content of Section 3 is included solely for the convenience of the reader. On the other hand, Theorem B is a substantial improvement over its counterpart [Sal19, Theorem 9.5]. The basic outline is the same, but many of the constituent arguments have been sharpened and simplified. The reader who is primarily interested in the theory of the stabilizer groups $\mathrm{Mod}_g[\phi]$ is encouraged to treat Theorem B as the "canonical" version, and is referred to [Sal19, Theorem 9.5] only as necessary. For a more detailed discussion of the proof of Theorem B, see the outline given just below.

Outline of Theorem A. The outline of the proof of Theorem A essentially follows that of its counterpart [Cal19, Theorem 1.1]. In Definition 3.6, we introduce the "geometric monodromy group" $\mathcal{G}(\mathcal{H}) \leqslant \mathrm{Mod}(\Sigma_g)$ of a (connected component of a) stratum $\mathcal{H}$ of unmarked abelian differentials and show in Proposition 3.7 that the classification of components of strata of marked differentials reduces to the problem of determining $\mathcal{G}(\mathcal{H})$. In Section 3.3, we give a construction of a square–tiled surface in each stratum using the method of Thurston and Veech; this surface is constructed so as to have a set of cylinders in correspondence with the Dehn twist generators described in Theorem B. Each such cylinder gives rise to a Dehn twist in $\mathcal{G}(\mathcal{H})$ (Lemma 3.9), so Theorem B implies that this collection of Dehn twists causes $\mathcal{G}(\mathcal{H})$ to be "as large as possible," leading to the classification of components of strata.

Outline of Theorem B. The proof of Theorem B in turn largely follows the outline of the proof of [Sal19, Theorem 9.5], with one modification that allows for a cleaner argument with less casework. The result of [Sal19, Theorem 9.5] did not treat the maximal case $r = 2g-2$, but here we are able to do so. In fact, we find that the case of general r described in Theorem B.3 follows very quickly from the maximal case (see Section 6.5). Accordingly, the bulk of the proof only treats the case $r = 2g-2$.

The argument in the case $r = 2g-2$ proceeds in two stages. The first stage, presented in Section 5 as Proposition 5.1, shows that the finite collections of twists given in Theorem B.1/2 generates an intermediate subgroup $\mathcal{T}_\phi \leqslant \mathrm{Mod}_g[\phi]$, the subgroup of "admissible twists" (see Definition 2.6). This is the group generated by Dehn twists about "admissible curves" (again see Definition 2.6).

The set of admissible curves determine a subgraph of the curve graph, and Proposition 5.1 is proved by working one's way out in this complex, using combinations of admissible twists to "acquire" twists about curves further and further out in the complex. This is encapsulated in Lemma 5.4 (note that the connection with curve complexes is all contained within the proof of Lemma 5.4, which is imported directly from [Sal19]). The corresponding arguments in [Sal19] made use of the existence of a certain configuration of curves which does not exist when $r \geq g-1$. Here we avoid this issue by directly showing that the configurations of Theorem B have the requisite properties (c.f. Lemmas 5.10, 5.13).

The second step is to show that the admissible subgroup $\mathcal{T}_\phi$ coincides with the stabilizer $\mathrm{Mod}_g[\phi]$, an *a priori* larger group. This result appears as Proposition 6.1; the proof takes place in Section 6. The method here is to show that both $\mathcal{T}_\phi$ and $\mathrm{Mod}_g[\phi]$ have the same intersection with the "Johnson filtration" on $\mathrm{Mod}(\Sigma_g)$ (c.f. Section 6.1). The outline exactly mirrors its counterpart in [Sal19]: Proposition 6.1 follows by assembling the three Lemmas 6.3, 6.4, 6.5, each of which shows that $\mathcal{T}_\phi$ and $\mathrm{Mod}_g[\phi]$ behave identically with respect to a certain piece of the Johnson filtration. The arguments provided here are both sharper and in many cases simpler than their predecessors in [Sal19]. In particular, the previous version of Lemma 6.4 required an intricate lower bound on genus which we replace here with the uniform (and optimal) requirement $g \geq 3$. The other main result of this section, Lemma 6.5, also improves on its predecessor. The previous version of Lemma 6.5 was only applicable for r odd, but here we are able to treat arbitrary r. The internal workings of this step have also been improved and are now substantially less coordinate–dependent.

Prior to the work carried out in Sections 5 and 6, in Section 4 we prove a lemma known as the "sliding principle" (Lemma 4.4). This is a flexible tool for carrying out computations involving the action of Dehn twists on curves, and largely subsumes the work done in [Cal19, Appendix A]. We believe that the sliding principle will be widely applicable to the study of the mapping class group.

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Contents

2. Higher spin structures

Theorem A asserts that the non–hyperelliptic components of strata of marked abelian differentials are classified by an object known as an "r–spin structure." Here we introduce the basic theory of such objects. After defining spin structures and their stabilizer subgroups in Section 2.1, we explain how r–spin structures arise from vector fields in Section 2.2. In Section 2.3, we connect the theory of r–spin structures to the classical theory of spin structures and quadratic forms on vector spaces in characteristic 2.

2.1. Basic properties.

There are several points of view on r–spin structures: they can be defined algebro–geometrically as a root of a line bundle, topologically as a cohomology class, or as an invariant of isotopy classes of simple closed curves on surfaces. For a more complete discussion, including proofs of the claims below, see [Sal19, Section 3]. In this work we only need to study r–spin structures from the point of view of surface topology; this approach is originally due to Humphries and Johnson [HJ89].

Definition 2.1 (r–spin structure)**.** Let Σ_g be a closed surface of genus $g \geq 2$, and let $\mathcal{S}$ denote the set of isotopy classes of oriented simple closed curves on Σ_g; we include here the inessential curve ζ that bounds an embedded disk to its left. An *r–spin structure* is a function $\phi : \mathcal{S} \to \mathbb{Z}/r\mathbb{Z}$ satisfying the following two properties:

(1) (Twist–linearity) Let $c, d \in \mathcal{S}$ be arbitrary. Then

$$\phi(T_c(d)) = \phi(d) + \langle d, c\rangle \phi(c) \pmod{r},$$

where $\langle c, d\rangle$ denotes the algebraic intersection pairing.

(2) (Normalization) For ζ as above, $\phi(\zeta) = 1$.

Remark 2.2. It can be shown (e.g. using homological coherence (Lemma 2.3 below)) that r must divide $2g - 2$.

An essential fact about r–spin structures is that they behave predictably on collections of curves bounding an embedded subsurface. This property is called *homological coherence.*

Lemma 2.3 (Homological coherence)**.** *Let ϕ be an r–spin structure on Σ_g, and let $S \subset \Sigma_g$ be a subsurface. Suppose $\partial S = c_1 \cup \cdots \cup c_k$ and all boundary components c_i are oriented so that S lies to the left. Then*

$$\sum_{i=1}^{k} \phi(c_i) = \chi(S).$$

Homological coherence quickly implies that a given spin structure is determined by its set of values on a basis for homology. Moreover, Lemma 2.4 stated below shows that the converse is true as well. In preparation, we define a *geometric homology basis* $\mathcal{B} = \{x_1, \ldots, x_{2g}\}$ to be a collection of oriented simple closed curves whose homology classes are linearly independent and generate $H_1(\Sigma_g; \mathbb{Z})$.

Lemma 2.4 (r–spin structures and geometric homology bases)**.** *Let*

$$\mathcal{B} = \{x_1, \ldots, x_{2g}\}$$

be a geometric homology basis. If ϕ, ψ are two r–spin structures on Σ_g such that $\phi(x_i) = \psi(x_i)$ for $1 \leqslant i \leqslant 2g$, then $\phi = \psi$.

Conversely, given $\mathcal{B}$ as above and any vector $v = (v_i) \in (\mathbb{Z}/r\mathbb{Z})^{2g}$, there exists an r–spin structure ϕ such that $\phi(x_i) = v_i$ for $1 \leqslant i \leqslant 2g$.

There is an action of the mapping class group $\mathrm{Mod}(\Sigma_g)$ on the set of r–spin structures: for $f \in \mathrm{Mod}(\Sigma_g)$ and $c \in \mathcal{S}$, define $(f \cdot \phi)(c) = \phi(f^{-1}(c))$.

Definition 2.5 (Stabilizer subgroup)**.** Let ϕ be a spin structure on a surface Σ_g. The *stabilizer subgroup* of ϕ, written $\mathrm{Mod}_g[\phi]$, is defined as

$$\mathrm{Mod}_g[\phi] = \{f \in \mathrm{Mod}(\Sigma_g) \mid (f \cdot \phi) = \phi\}.$$

The simplest class of elements of $\mathrm{Mod}_g[\phi]$ are the Dehn twists that preserve ϕ. By twist–linearity (Definition 2.1.1), if c is a nonseparating curve, T_c preserves ϕ if and only if $\phi(c) = 0$.

Definition 2.6 (Admissible twist, admissible subgroup)**.** Let ϕ be an r–spin structure on Σ_g. A nonseparating simple closed curve c is said to be *ϕ–admissible* if $\phi(c) = 0$ (if the spin structure ϕ is implied, it will be omitted from the notation). The corresponding Dehn twist $T_c \in \mathrm{Mod}_g[\phi]$ is called an *admissible twist.* The subgroup

$$\mathcal{T}_\phi = \langle T_c \mid \phi(c) = 0,\ c \text{ nonseparating}\rangle \leqslant \mathrm{Mod}_g[\phi]$$

is called the *admissible subgroup*.

Remark 2.7. In general, the value $\phi(c)$ depends on the orientation of c. However, if c is given the opposite orientation then $\phi(c)$ changes sign, so admissibility is a property of *unoriented* curves.

2.2. **Spin structures from winding number functions.** The spin structures under study in this paper arise from a construction known as a "winding number function" originally due to Chillingworth [Chi72]. We sketch here the basic idea; see [HJ89] for details.

Example 2.8 (Winding number function)**.** Let Σ_g be a compact surface endowed with a vector field V with isolated zeroes $p_1, \dots, p_n$ of orders $k_1, \dots, k_n$. Suppose

$$\gamma : S^1 \to \Sigma_g \setminus \{p_1, \dots, p_k\}$$

is a C^1–embedded curve on $\Sigma_g \setminus \{p_1, \dots, p_n\}$. Then the winding number of the tangent vector $\gamma'(t)$ relative to $V(\gamma(t))$ determines a $\mathbb{Z}$–valued winding number for γ. As γ passes over a zero of order k_i, the winding number of γ changes by a value of k_i. Thus, if $r = \gcd(k_1, \dots, k_n)$, the function

$$wn_V : \mathcal{S} \to \mathbb{Z}/r\mathbb{Z}$$

is a well–defined map from the set of isotopy classes of oriented curves to $\mathbb{Z}/r\mathbb{Z}$. Both twist–linearity and the fact that $\phi(\zeta) = 1$ are easy to check, so in fact wn_V is an r–spin structure.

Accordingly, we sometimes speak of the value $\phi(c)$ as the "winding number" of c even when ϕ does not manifestly arise from this construction.

2.3. **Classical spin structures and the Arf invariant.** If r is even, then the mod 2 reduction of ϕ determines a "classical" spin structure. A basic understanding of the special features present in this case is necessary for a full understanding of r–spin structures for $r > 2$ even. In Lemma 2.9 we note the basic fact that bridges our notion of a 2–spin structure with the classical formulation via quadratic forms. We then proceed to define the "Arf invariant" (Definition 2.12) and recall some of its basic properties.

From 2–spin structures to quadratic forms. For $r > 2$, the value of ϕ on a simple closed curve c depends on c itself, and not merely the homology class $[c] \in H_1(\Sigma_g; \mathbb{Z})$. However, the information encoded in a 2–spin structure is "purely homological":

Lemma 2.9. *Let ϕ be a 2–spin structure on Σ_g and let $c \subset \Sigma_g$ be a simple closed curve. Then $\phi(c) \in \mathbb{Z}/2\mathbb{Z}$ depends only on the homology class $[c] \in H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$.*

Following Lemma 2.9, if ϕ is an r–spin structure for $r > 2$ even, we define the mod 2 value of ϕ on a homology class $z \in H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$ to be $\phi(c) \pmod 2$ for any simple closed curve c with $[c] = z$. This gives rise to an algebraic structure on $H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$ known as a *quadratic form*. In preparation, recall that if V is a vector space over a field of characteristic 2, a *symplectic form* $\langle \cdot, \cdot \rangle$ is defined to be a bilinear form satisfying $\langle v, v \rangle = 0$ for all $v \in V$.

Definition 2.10. Let V be a vector space over $\mathbb{Z}/2\mathbb{Z}$ equipped with a symplectic form $\langle \cdot, \cdot \rangle$. A *quadratic form* q on V is a function $q : V \to \mathbb{Z}/2\mathbb{Z}$ satisfying

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle.$$

Remark 2.11. There is a standard correspondence between 2–spin structures and quadratic forms which generalizes for any even $r \geq 2$. If ϕ is an r–spin structure for $r \geq 2$ even, then the function

$$q(x) = \phi(x) + 1 \pmod 2$$

is a quadratic form on $H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$; here one evaluates $\phi(x)$ on $x \neq 0$ by choosing a simple closed curve representative for x and applying Lemma 2.9.

Orbits of quadratic forms and the Arf invariant. The symplectic group $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ acts on the set of quadratic forms on $H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$ by pullback. Here we recall the *Arf invariant* which describes the orbit structure of this group action.

Definition 2.12 (Arf invariant)**.** Let V be a vector space over $\mathbb{Z}/2\mathbb{Z}$ equipped with a symplectic form $\langle \cdot, \cdot \rangle$, and q be a quadratic form on V. The *Arf invariant* of q, written $\mathrm{Arf}(q)$, is the element of $\mathbb{Z}/2\mathbb{Z}$ defined by

$$\mathrm{Arf}(q) = \sum_{i=1}^{g} q(x_i) q(y_i) \pmod 2,$$

where $\{x_1, y_1, \ldots, x_g, y_g\}$ is *any* symplectic basis for V.

For an r–spin structure ϕ on Σ_g with $r \geq 2$ even, $\mathrm{Arf}(\phi)$ is defined to be the Arf invariant of the quadratic form associated to ϕ by Remark 2.11.

A quadratic form q is said to be *even* or *odd* according to the parity of $\mathrm{Arf}(q)$. The parity of an r–spin structure for $r \geq 2$ even is defined analogously.

The Arf invariant of a spin structure is easy to compute given any collection of curves which span the homology of the surface. We say that a *geometric symplectic basis* for Σ_g is a collection

$$\mathcal{B} = \{x_1, y_1, \ldots, x_g, y_g\}$$

of $2g$ curves on S such that $i(x_i, y_i) = 1$ for $i = 1, \ldots, g$, and such that all other intersections are zero (here $i(c, d)$ denotes the *geometric intersection number* of c, d). Then $\mathrm{Arf}(\phi)$ may be computed as

$$\mathrm{Arf}(\phi) = \sum_{i=1}^{g} (\phi(x_i) + 1)(\phi(y_i) + 1) \pmod 2,$$

where $\mathcal{B} = \{x_1, y_1, \ldots, x_g, y_g\}$ is *any* geometric symplectic basis on Σ_g.

Remark 2.13. The Arf invariant is additive under direct sum; that is, if $V = W_1 \oplus W_2$ where W_1 and W_2 are symplectically orthogonal and are equipped with nondegenerate quadratic forms q_1 and q_2, then one has

$$\mathrm{Arf}(q_1 \oplus q_2) = \mathrm{Arf}(q_1) + \mathrm{Arf}(q_2).$$

If $S \subset \Sigma_g$ is a subsurface with one boundary component, then the r–spin structure ϕ admits an obvious restriction to an r–spin structure $\phi\,|_S$ on S. In this way we speak of the Arf invariant of a subsurface S, i.e. $\mathrm{Arf}(\phi\mid_S)$. If $\Sigma_g = S_1 \cup S_2$ where both subsurfaces have a single boundary component, then the Arf invariant is additive in the obvious sense. The Arf invariant is not defined in any straightforward way on a surface with 2 or more boundary components.

Since 2–spin structures (or equivalently, quadratic forms on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$) are "purely homological" in the sense of Lemma 2.9, the action of the mapping class group on the set of 2–spin structures factors through the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $H_1(\Sigma_g;\mathbb{Z})$ and ultimately through $\mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$ acting on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$. Thus there is an algebraic counterpart to the notion of spin structure stabilizer defined in Definition 2.5.

Definition 2.14 (Algebraic stabilizer subgroup)**.** Let q be a quadratic form on $H_1(\Sigma_g;\mathbb{Z}/2\mathbb{Z})$. The *algebraic stabilizer subgroup* is the subgroup

$$\mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z})[q] = \{A \in \mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z}) \mid A\cdot q = q\}.$$

We define the algebraic stabilizer subgroup $\mathrm{Sp}(2g,\mathbb{Z})[q]$ as the preimage of $\mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z})[q]$ in $\mathrm{Sp}(2g,\mathbb{Z})$.

The Arf invariant of a quadratic form is invariant under the action of $\mathrm{Sp}(2g,\mathbb{Z}/2\mathbb{Z})$ (and hence under the action of $\mathrm{Sp}(2g,\mathbb{Z})$ and $\mathrm{Mod}(\Sigma_g)$) [Arf41], and in fact this is the only invariant of the $\mathrm{Mod}(\Sigma_g)$ action.

More generally, for any even r the $\mathrm{Mod}(\Sigma_g)$ action on the set of r–spin structures must always preserve the induced Arf invariant, and as above, this is the only invariant of the $\mathrm{Mod}(\Sigma_g)$ action.

Lemma 2.15 (c.f. Propositions 4.2 and 4.9 in [Sal19])**.** *Let Σ_g be a closed surface of genus $g \geq 2$ and let r divide $2g-2$. If r is odd, then the mapping class group acts transitively on the set of r–spin structures. If r is even, then there are two orbits of the* $\mathrm{Mod}(\Sigma_g)$ *action, distinguished by their Arf invariant.*

Consequently, if ϕ is an r–spin structure, then the index $[\mathrm{Mod}(\Sigma_g) : \mathrm{Mod}_g[\phi]]$ *is*

- r^{2g} *if r is odd,*
- $(r/2)^{2g}\left(2^{g-1}(2^g+1)\right)$ *if r is even and ϕ has even Arf invariant, and*
- $(r/2)^{2g}\left(2^{g-1}(2^g-1)\right)$ *if r is even and ϕ has odd Arf invariant.*

3. Theorem A: the classification of connected components

In this section, we prove Theorem A assuming Theorem B. Our strategy matches the one employed by the first author in [Cal19, §§5,6], but we reproduce the details below for the convenience of the reader.

The plan of proof is as follows: appealing to Kontsevich and Zorich's classification of the components of $\Omega\mathcal{M}(\underline{\kappa})$ (Theorem 3.5), we equate the classification of components of $\Omega\mathcal{T}(\underline{\kappa})$ with the computation of the geometric monodromy groups (see Definition 3.6) of components of $\Omega\mathcal{M}(\underline{\kappa})$. This is recorded as Proposition 3.7.

Ultimately we prove that each geometric monodromy group *coincides* with the stabilizer of some r-spin structure, and so by the orbit–stabilizer theorem, r–spin structures are in 1–to–1 correspondence with the (non–hyperelliptic) components of $\Omega\mathcal{T}(\underline{\kappa})$. Lemma 3.8 shows that each geometric monodromy group stabilizes some r–spin structure ϕ, demonstrating one direction of inclusion.

The reverse inclusion follows by applying Theorem B. In Section 3.3, we use a construction of Thurston and Veech to build an abelian differential out of a system of curves satisfying the hypotheses of Theorem B (Lemma 3.14). The geometry of these differentials allows us to realize the Dehn twist in each curve as a continuous deformation of the flat structure, thereby implying that every element of $\mathrm{Mod}_g[\phi]$ is realized through flat deformations and so proving Theorem A.

3.1. **The flat geometry of an abelian differential.** We begin with a review of some background information on abelian differentials and their induced flat cone metrics. While not every flat cone metric comes from an abelian differential (Lemma 3.1), those that do induce r–spin structures (Construction 3.2). In Lemma 3.4, we record a first result relating the geometry of the flat metric with the admissible curves for the induced spin structure.

An *abelian differential* ω on a Riemann surface X is a holomorphic section of the canonical bundle K_X. In charts away from its zeros $\{p_1, \ldots, p_n\}$, the form is locally equivalent to dz, while at each p_i it is locally equivalent to $z^{k_i} dz$ for some $k_i \geq 1$. The metric given by $|dz|^2$ is then Euclidean away from each p_i, at which the metric has a cone angle of $2\pi(k_i + 1)$. Along with the flat cone metric, ω also induces a "horizontal" vector field $V = 1/\omega$ away from $\{p_1, \ldots, p_n\}$, at which V has index $-k_i$. For a more thorough discussion, see, e.g., [Zor06].

In practice, it is often useful not to build abelian differentials directly, but instead to build flat metrics and then check that they are induced from abelian differentials. For any locally flat metric σ on a closed surface Σ_g with finitely many cone points $p_1, \ldots, p_n$, there is a natural *holonomy representation*

$$\mathrm{hol}_\sigma : \pi_1(\Sigma \setminus \{p_1, \ldots, p_n\}) \to SO(2)$$

which measures the rotational difference between a tangent vector and its parallel transport along a loop in $\Sigma \setminus \{p_1, \ldots, p_n\}$.

It is a standard fact that the holonomy of a locally flat metric determines whether or not it comes from an abelian differential (see, e.g., [MT02, §1.8]).

Lemma 3.1. *Let σ be a flat cone metric on a closed surface Σ_g with cone points $\{p_1, \ldots, p_n\}$. Then the following are equivalent:*

(1) There exists some Riemann surface X and abelian differential ω so that (X, ω) with the induced metric $|dz|^2$ is isometric to (Σ, σ)
(2) The holonomy representation $hol_\sigma : \pi_1(\Sigma \setminus \{p_1, \ldots, p_n\}) \to SO(2)$ is trivial.
(3) The cone angle at each p_i is $2\pi(k_i + 1)$ for some $k_i \in \mathbb{N}$ and there exists a locally constant vector field V on (Σ, σ), singular only at the p_i with index $-k_i$ at each p_i.

A vector field V as above is sometimes called a *translation vector field.* The winding number with respect to a translation vector field serves as our bridge between the flat geometry of an abelian differential and its induced r–spin structure.

Construction 3.2 (c.f. Example 2.8)**.** Suppose that X is a Riemann surface equipped with an abelian differential ω with zeros of order $(k_1, \ldots, k_n)$. Let $V = 1/\omega$. Then the winding number function wn_V of a curve with respect to V defines an r–spin structure ϕ on X, where $r = \gcd(k_1, \ldots, k_n)$.

If X is also equipped with a *marking*, that is, an isotopy class of map $f : \Sigma \to X$ where Σ_g is a reference topological surface, then ϕ pulls back to an r–spin structure on Σ_g, which by abuse of notation we will also denote by ϕ. In this way, we see that any marked abelian differential (X, f, ω) gives rise to an r–spin structure on Σ_g.

Remark 3.3. It is not hard to see that any two translation vector fields on (X, ω) are related by a rotation, and so any two translation vector fields will induce the same r–spin structure as $V = 1/\omega$.

Because of the relationship between the flat geometry of (X, ω) and the vector field V, it is easy to produce examples of curves on X of winding number 0. If Σ_g is a surface equipped with a flat cone metric σ, then a *cylinder* on (Σ, σ) is an embedded Euclidean cylinder which contains no cone points in its interior.

Lemma 3.4 (Lemma 4.6 of [Cal19])**.** *If (X, ω) is an abelian differential defining an r–spin structure ϕ, then the core curve of any cylinder on (X, ω) is ϕ–admissible.*

3.2. **Geometric monodromy and cylinder twists.** Lemma 3.4 provides a first point of contact between the flat geometry of an abelian differential and the algebra of a spin structure stabilizer. In this subsection, we deepen this link, recording in Theorem 3.5 Kontsevich and Zorich's classification of components of strata over moduli space. We then show in Lemma 3.8 that *any* deformation of a marked abelian differential preserves the associated r–spin structure. In Lemma 3.9, we begin the process of establishing the converse (and hence Theorem A) by showing that each cylinder gives rise to an admissible twist.

The relationship between abelian differentials and spin structures is not new; indeed, it plays a pivotal role in Kontsevich and Zorich's classification of the components of strata of unmarked differentials. In the absence of a marking, it is impossible to compare the spin structures induced by two points (X, ω) and (X', ω'). It is therefore the Arf invariant, not the spin structure, which serves as a classifying invariant for the (non-hyperelliptic) components of $\Omega\mathcal{M}(\underline{\kappa})$.

Theorem 3.5 (Theorem 1 of [KZ03])**.** *Let $g \geq 4$ and $\underline{\kappa} = (k_1, \ldots, k_n)$ be a partition of $2g-2$. Then the components of $\Omega\mathcal{M}(\underline{\kappa})$ are classified as follows:*

- *If $\gcd(\underline{\kappa}) \in \{2g-2, g-1\}$, then $\Omega\mathcal{M}(\underline{\kappa})$ has a unique hyperelliptic connected component.*
- *If $\gcd(\underline{\kappa})$ is odd, then $\Omega\mathcal{M}(\underline{\kappa})$ has a unique non–hyperelliptic connected component.*
- *If $\gcd(\underline{\kappa})$ is even, then $\Omega\mathcal{M}(\underline{\kappa})$ has exactly two non–hyperelliptic components, distinguished by the Arf invariant of the 2–spin structure induced by (X, ω).*

With this classification in hand, we can now record how deformations of an abelian differential give rise to mapping classes. Note that each component of $\Omega\mathcal{M}(\underline{\kappa})$ is generally not a manifold but an orbifold, which is locally modeled on $H^1(X, \mathrm{Zeros}(\omega); \mathbb{C})$ away from its singularities (see, e.g., [Wri16, Lemma 2.1]).

Definition 3.6. Let $\mathcal{H}$ be a component of $\Omega\mathcal{M}(\underline{\kappa})$ and let $(X, \omega) \in \mathcal{H}$ be a generic (non-orbifold) point. Any loop γ in $\mathcal{H}$ induces a(n isotopy class of) self–diffeomorphism $\rho(\gamma)$ of X by parallel transport.

The *geometric monodromy group* $\mathcal{G}(\mathcal{H})$ is the image of the map

$$\rho : \pi_1^{\mathrm{orb}}(\mathcal{H}, (X,\omega)) \to \mathrm{Mod}(X).$$

Fixing a marking $f : \Sigma_g \to X$ identifies $\mathcal{G}(\mathcal{H})$ as a subgroup of $\mathrm{Mod}(\Sigma_g)$, but choosing a different basepoint (X,ω) or different marking f will conjugate $\mathcal{G}(\mathcal{H})$ in $\mathrm{Mod}(\Sigma_g)$. As such, in the following statements we will always begin by fixing some marked abelian differential (X, f, ω) with $(X,\omega) \in \mathcal{H}$; then the geometric monodromy group should be understood to be defined with reference to basepoint (X,ω) and marking f.

Proposition 3.7. *Let $\mathcal{H}$ be a component of $\Omega\mathcal{M}(\underline{\kappa})$ and fix (X, f, ω) with $(X,\omega) \in \mathcal{H}$. Then the components of $\Omega\mathcal{T}(\underline{\kappa})$ which cover $\mathcal{H}$ are in bijective correspondence with the cosets of $\mathcal{G}(\mathcal{H})$ in $\mathrm{Mod}(\Sigma_g)$.*

Proof. Let $\widetilde{\mathcal{H}}$ be the component of $\Omega\mathcal{T}(\underline{\kappa})$ containing (X, f, ω). The mapping class group acts on the set of components of $\Omega\mathcal{T}(\underline{\kappa})$ which cover $\mathcal{H}$ by permutations, and so it suffices to show that

$$\mathcal{G}(\mathcal{H}) = \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(\widetilde{\mathcal{H}}).$$

Now if a mapping class g is in $\mathcal{G}(\mathcal{H})$ then it is the image of a loop γ in $\Omega\mathcal{M}(\underline{\kappa})$, which can be lifted to a path $\widetilde{\gamma}$ in $\Omega\mathcal{T}(\underline{\kappa})$ from (X, f, ω) to $g \cdot (X, f, \omega)$. Therefore $g \in \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(\widetilde{\mathcal{H}})$.

Conversely, if g stabilizes $\widetilde{\mathcal{H}}$, then since $\widetilde{\mathcal{H}}$ is path–connected there is a path $\widetilde{\gamma}$ in $\widetilde{\mathcal{H}}$ from (X, f, ω) to $g \cdot (X, f, \omega)$. The projection of $\widetilde{\gamma}$ to $\mathcal{H}$ is a loop γ whose geometric monodromy is exactly g, and hence $g \in \mathcal{G}(\mathcal{H})$. □

Because the horizontal vector field of (X,ω) deforms continuously along with (X,ω), the winding number of any curve on X is constnat and so the geometric monodromy group must preserve the induced r–spin structure.

Lemma 3.8 (Corollary 4.8 in [Cal19])**.** *Let $g \geq 2$ and $\underline{\kappa}$ a partition of $2g-2$ with $\gcd(\underline{\kappa}) = r$. Let $\mathcal{H}$ be a component of $\Omega\mathcal{M}(\underline{\kappa})$ and fix (X, f, ω) with $(X,\omega) \in \mathcal{H}$. Then*

$$\mathcal{G}(\mathcal{H}) \leqslant \mathrm{Mod}_g[\phi],$$

where ϕ is the r–spin structure corresponding to (X, f, ω).

To exhibit the reverse inclusion, we need a way to build elements of $\mathcal{G}(\mathcal{H})$. A particularly simple method is to realize curves as cylinders; then the corresponding Dehn twists can be realized as continuous deformations of flat surfaces, and hence as elements of $\mathcal{G}(\mathcal{H})$.

Lemma 3.9 (c.f. Lemma 6.2 in [Cal19])**.** *Let $\mathcal{H}$ be a component of $\Omega\mathcal{M}(\underline{\kappa})$ and fix (X, f, ω) with $(X,\omega) \in \mathcal{H}$. If c is a simple closed curve on Σ_g such that $f(c)$ is the core curve of a cylinder on (X,ω), then $T_c \in \mathcal{G}(\mathcal{H})$.*

Remark 3.10. Lemma 6.2 of [Cal19] only deals with the case when $\mathcal{H}$ is a non-hyperelliptic component of $\Omega\mathcal{M}(\underline{\kappa})$. When $\mathcal{H}$ consists entirely of hyperelliptic differentials, the result follows from the description of $\mathcal{G}(\mathcal{H})$ appearing in the proof of [Cal19, Corollary 2.6] together with the fact that the hyperelliptic involution of any $(X,\omega) \in \mathcal{H}$ (setwise) fixes each of its cylinders (see, e.g., [Lin15, Lemma 2.1]).

3.3. **Construction of prototypes and the proof of the classification theorem.** We now recall the Thurston–Veech method for building a flat surface out of a filling pair of multicurves. In Lemma 3.12, we use this procedure to build a locally flat metric with a collection of cylinders whose core curves satisfy the hypotheses of Theorem B. In Lemma 3.14, we analyze the holonomy of this metric in order to show that this is induced from an abelian differential. This is used to deduce that the geometric monodromy group of a component of $\Omega\mathcal{M}(\underline{\kappa})$ is exactly the stabilizer of the corresponding r–spin structure, completing the proof of Theorem A.

Definition 3.11 (Definition 5.1 of [Cal19])**.** Suppose that $g \geq 3$ and let $\underline{\kappa} = (k_1, \ldots, k_n)$ be a partition of $2g-2$. If $\gcd(\underline{\kappa})$ is even, also choose $\mathrm{Arf} \in \{0,1\}$. Label curves of Σ_g as follows:

(1) If $\gcd(\underline{\kappa})$ is odd, then label the curves as in Figure 3a
(2) If $\gcd(\underline{\kappa})$ is even and
$$\mathrm{Arf}(\phi) = \begin{cases} 1 & g \equiv 0,3 \pmod 4 \\ 0 & g \equiv 1,2 \pmod 4 \end{cases}$$
then label the curves as in Figure 3a.
(3) If $\gcd(\underline{\kappa})$ is even and
$$\mathrm{Arf}(\phi) = \begin{cases} 1 & g \equiv 1,2 \pmod 4 \\ 0 & g \equiv 0,3 \pmod 4 \end{cases}$$
then label the curves as in Figure 3b.

No matter the labeling scheme, define the curve system $\mathsf{C}(\underline{\kappa}, \mathrm{Arf})$ to be the collection
$$\mathsf{C}(\underline{\kappa}, \mathrm{Arf}) = \{a_i\}_{i=1}^{2g-1} \cup \left\{ b_i : i = \sum_{j=1}^{\ell} k_j \text{ for } \ell = 1, \ldots, n \right\},$$
where the b_i indices are understood mod $2g-2$.

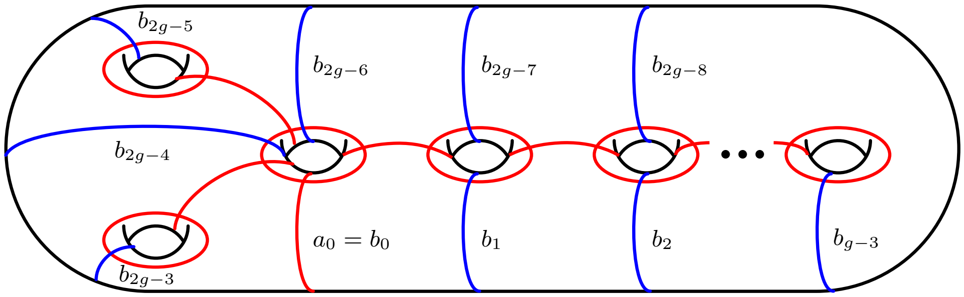
We first see that $\mathsf{C}(\underline{\kappa}, \mathrm{Arf})$ determines a flat surface.

Lemma 3.12. *Suppose that $g \geq 3$. Let $\underline{\kappa}$ be any partition $\underline{\kappa}$ of $2g-2$, and if $\gcd(\underline{\kappa})$ is even, choose $\mathrm{Arf} \in \{0,1\}$. Then there exists a flat cone metric σ on Σ_g such that the curves of the curve system $\mathsf{C}(\underline{\kappa}, \mathrm{Arf})$ are realized as cylinders on (Σ, σ).*

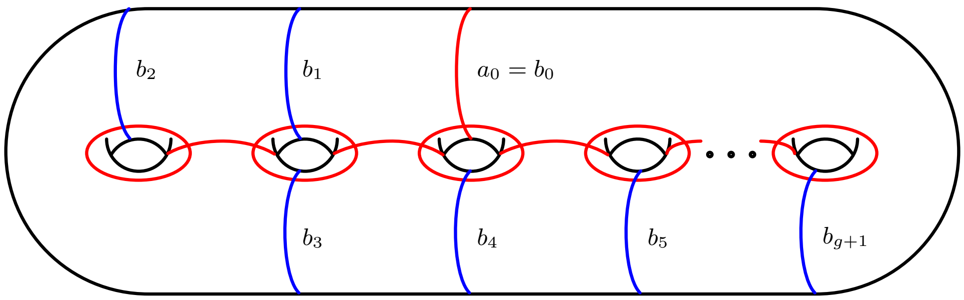
Proof. This is nothing more than an application of the Thurston–Veech construction; we recall the details for the interested reader below.

Suppose that γ_h and γ_v are any pair of multicurves which jointly fill Σ_g. Their union therefore defines a cellulation of Σ_g whose 0–cells are the intersection points of γ_h with γ_v, whose 1–cells are the simple arcs of $\gamma_h \cup \gamma_v$, and the 2–cells of which are n polygonal disks with $2(m_1+2)$, …, $2(m_n+2)$ sides, respectively.

The dual complex is therefore built out of (topological) squares, with $2(m_i+2)$ of them meeting around the i^{th} vertex. Declaring each square to be a flat unit square yields a flat cone metric σ on Σ_g with cone angle
$$\frac{\pi}{2} \cdot 2(m_i+2) = m_i + 2$$



(A) Labels in cases (1) and (2) of Definition 3.11.



(B) Labels in case (3) of Definition 3.11.

FIGURE 3. Naming conventions for simple closed curves, depending on $\gcd(\underline{\kappa})$, Arf, and g.

around the i^{th} cone point p_i.

The intersection graph associated to the curve system $\mathsf{C}(\underline{\kappa}, \mathrm{Arf})$ is a tree, and therefore the curves can be partitioned into two multicurves γ_h and γ_v. We call the flat cone metric $\sigma = \sigma(\underline{\kappa}, \mathrm{Arf})$ resulting from the Thurston–Veech construction the *prototype* for the pair $(\underline{\kappa}, \mathrm{Arf})$. □

Remark 3.13. Observe that by construction, σ has cone points of angles

$$2(k_1 + 1)\pi, \dots, 2(k_n + 1)\pi,$$

and if we assume that σ does indeed come from an abelian differential ω (as established in Lemma 3.14), then the Arf invariant of the r–spin structure induced by ω agrees with the choice of $\mathrm{Arf} \in \{0, 1\}$ used to construct it [Cal19, Lemma 5.4].

Lemma 3.14. *Suppose that $g \geq 3$. Let $\underline{\kappa}$ be any partition $\underline{\kappa}$ of $2g-2$, and if $\gcd(\underline{\kappa})$ is even, choose $\mathrm{Arf} \in \{0, 1\}$ (if $g = 3$, set $\mathrm{Arf} = 1$). Then there exists a non-hyperelliptic marked abelian differential (X, f, ω) in $\Omega\mathcal{T}(\underline{\kappa})$ such that the curves of the curve system $\mathsf{C}(\underline{\kappa}, \mathrm{Arf})$ are realized as the vertical and horizontal cylinders of (X, f, ω).*

Note that in the case $g = 3$, the components of $\Omega\mathcal{M}(4)$ and $\Omega\mathcal{M}(2, 2)$ with $\mathrm{Arf} = 0$ coincide with the hyperelliptic components [KZ03, Theorem 2].

Proof. In order to prove that the metric σ is induced from an abelian differential, we show that our prototype surface admits a translation vector field V outside of the singularities. Therefore, by Lemma 3.1, there is some $(X, f, \omega) \in \Omega\mathcal{T}(\underline{\kappa})$ which is isometric to (Σ_g, σ) via a marking $f : \Sigma_g \to X$.

To build this vector field, we choose a positive horizontal direction on each square. If we can do this so that the squares glue consistently along each edge, we may then define V by pasting together the constant horizontal vector fields $\langle 1, 0\rangle$. The problem then becomes to find a coherent choice of positive horizontal direction for each square.

Observe that declaring the edges dual to the edges of γ_v to be horizontal and the edges dual to γ_h to be vertical naturally partitions the edges of the squares. Then the coherence condition on gluing squares is equivalent to the condition that the curves of γ_h and γ_v may be oriented so that each intersection of a curve of γ_h with one of γ_v is positively oriented. Now since the intersection graph of the multicurves γ_h and γ_v is a tree, one may choose the orientation of a single curve of γ_h and extend by the positivity constraint to yield a coherent orientation on γ_h and γ_v (see Figure 4). The choice of positive horizontal on each square induced from the orientation of γ_h then yields the desired result.

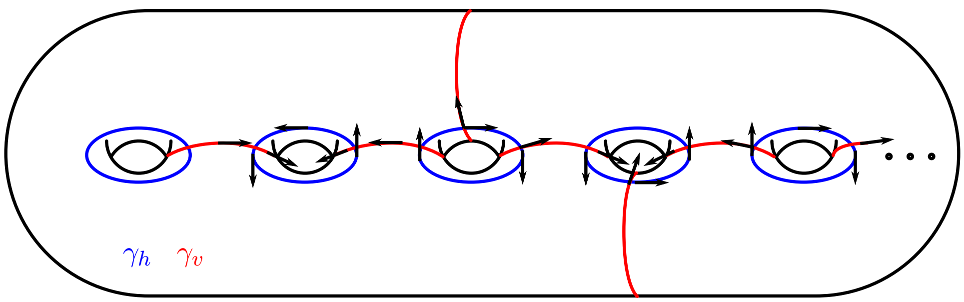


FIGURE 4. Extending the orientation of a single curve to a global orientation of $\gamma_h \cup \gamma_v$.

Suppose towards contradiction that (X, ω) is in a hyperelliptic component of $\Omega\mathcal{M}(\underline{\kappa})$; then by Theorem 3.5, we know that $\underline{\kappa} = (2g-2)$ or $(g-1, g-1)$. In this case, the hyperelliptic involution ι of (X, ω) must setwise fix each of its cylinders (see, e.g., [Lin15, Lemma 2.1]). In particular, ι fixes the curves $\{a_i\}_{i=1}^{2g-1}$ but reverses their orientation.

Therefore, each a_i curve is the lift of an arc α_i on X/ι connecting branch values of the associated cover $q : X \to X/\iota$. Now observe that (by the Birman–Hilden theory, see Section 4.2 or [MW]) the geometric intersection numbers of the a_i are determined by the intersection numbers of the α_i: indeed, one has

$$i(a_i, a_j) = 2i(\alpha_i, \alpha_j) + e(\alpha_i, \alpha_j)$$

where $i(\alpha_i, \alpha_j)$ counts only intersection points in the interior of the arcs and $e(\alpha_i, \alpha_j) \in \{0, 1, 2\}$ is the number of their shared endpoints. But now since

$$i(a_5, a_4) = i(a_5, a_6) = i(a_5, a_0) = 1$$

we know that α_5 shares an endpoint with each of $\{\alpha_4, \alpha_6, \alpha_0\}$. However, since α_5 has only two endpoints, this means that two of $\{\alpha_4, \alpha_6, \alpha_0\}$ share an endpoint, and hence their corresponding a_i curves intersect, a contradiction.

Therefore (X,ω) cannot be hyperelliptic. □

Since any twist in a cylinder of (X,f,ω) must stabilize the component of $\Omega\mathcal{T}(\underline{\kappa})$ in which it lies (Lemma 3.9), the classification theorem then follows from a quick application of Theorem B.

Proof of Theorem A. Let $g \geq 5$, let $\underline{\kappa}$ be a partition of $2g-2$, and choose $\mathcal{H}$ to be a non-hyperelliptic connected component of $\Omega\mathcal{M}(\underline{\kappa})$. Set $(X,f,\omega) \in \Omega\mathcal{T}(\underline{\kappa})$ to be the prototype for $(\underline{\kappa}, \mathrm{Arf}(\mathcal{H}))$ constructed in Lemma 3.14. Observe that $(X,\omega) \in \mathcal{H}$. Let ϕ be the r–spin structure induced by (X,f,ω), and define

$$\Gamma := \langle T_c : f(c) \text{ is a cylinder on } (X,\omega)\rangle .$$

By construction, the generating sets of cases 1 and 2 of Theorem B are realized as the core curves of cylinders on the appropriate prototype surface (X,ω); therefore

$$\mathrm{Mod}_g[\widetilde{\phi}] \leqslant \Gamma,$$

where $\widetilde{\phi}$ is some $(2g-2)$–spin structure that refines ϕ.

It remains to show that $\mathrm{Mod}_g[\phi] \leqslant \Gamma$. We begin by observing that for every i, the cut surface

$$\Sigma_g \setminus (b_0 \cup b_i \cup a_2 \cup a_4 \cup \ldots \cup a_{2g-2})$$

is the union of an $i+2$–holed and a $2g-i$–holed sphere, so homological coherence (Lemma 2.3) implies that each curve b_i has $\phi(b_i) = i$ (relative to an appropriate orientation). Therefore, by construction of the prototype (X,f,ω), we see that Γ contains twists on curves b_i with $\widetilde{\phi}$–values

$$\left\{\widetilde{\phi}(b_i)\right\} = \left\{\sum_{j=1}^{\ell} k_j : \ell = 1, \ldots, n\right\}.$$

Since $r = \gcd(\underline{\kappa})$, the set $\left\{\widetilde{\phi}(b_i)\right\}$ generates the subgroup $r\mathbb{Z}/(2g-2)\mathbb{Z}$ of $\mathbb{Z}/(2g-2)\mathbb{Z}$, and so Theorem B.3 implies that $\Gamma = \mathrm{Mod}_g[\phi]$.

Putting this together with Lemmas 3.9 and 3.8 yields

$$\mathrm{Mod}_g[\phi] = \Gamma \leqslant \mathcal{G}(\mathcal{H}) \leqslant \mathrm{Mod}_g[\phi] \tag{1}$$

and therefore all of the groups are equal. In particular, Proposition 3.7 together with Lemma 2.15 imply that there are exactly

$$[\mathrm{Mod}(\Sigma_g) : \mathrm{Mod}_g[\phi]] = \begin{cases} r^{2g} & \text{if } r \text{ is odd} \\ (r/2)^{2g}\left(2^{g-1}(2^g+1)\right) & \text{if } r \text{ is even and } \mathrm{Arf}(\mathcal{H}) = 0 \\ (r/2)^{2g}\left(2^{g-1}(2^g-1)\right) & \text{if } r \text{ is even and } \mathrm{Arf}(\mathcal{H}) = 1 \end{cases}$$

components of $\Omega\mathcal{T}(\underline{\kappa})$ lying over $\mathcal{H}$.

Combining the above statements for the components of $\Omega\mathcal{M}(\underline{\kappa})$, as classified by Theorem 3.5, completes the proof of the theorem. □

From this description of which deformations can occur in a stratum, we can also give a description of which curves appear as cylinders on a surface in a stratum.

Proof of Corollary 1.1. We first consider the case when $\widetilde{\mathcal{H}}$ is a non-hyperelliptic component of $\Omega\mathcal{T}(\underline{\kappa})$. Let ϕ denote the corresponding r–spin structure. Recall that we are trying to prove that a curve c is realized as the core curve of a cylinder on some marked abelian differential in $\widetilde{\mathcal{H}}$ if and only if it is nonseparating and ϕ–admissible.

Lemma 3.4 shows that the core curve of every cylinder on every $(X, f, \omega) \in \widetilde{\mathcal{H}}$ is ϕ–admissible, and by Stokes' theorem, no separating curve can ever be a cylinder on an abelian differential.

To see that the conditions are also sufficient, let (X, f, ω) be any marked abelian differential in $\widetilde{\mathcal{H}}$ (for example, the prototype coming from Lemma 3.14) and let ξ be a cylinder on (X, ω). Suppose that the core curve of ξ is $f(d)$, where d is a simple closed curve on Σ_g. By Lemma 3.4, d is ϕ–admissible.

As explained in Lemma 6.8, the spin stabilizer subgroup $\text{Mod}_g[\phi]$ acts transitively on the set of admissible curves, and hence there is some $g \in \text{Mod}_g[\phi]$ so that $g(d) = c$. Therefore, $f \circ g^{-1}(c) = f(d)$ is the core curve of ξ on (X, ω), and hence c is realized as the core curve of a cylinder on

$$g \cdot (X, f, \omega) = (X, fg^{-1}, \omega).$$

By Proposition 3.7, we have that (X, fg^{-1}, ω) is in $\widetilde{\mathcal{H}}$, finishing the proof.

Suppose now that $\widetilde{\mathcal{H}}$ is a hyperelliptic component of $\Omega\mathcal{T}(\underline{\kappa})$ with corresponding hyperelliptic involution ι; then as in the proof of Lemma 3.14, $\underline{\kappa} = (2g-2)$ or $(g-1, g-1)$ and ι fixes the core curves of each cylinder. Therefore, the core curve of each cylinder on any $(X, f, \omega) \in \widetilde{\mathcal{H}}$ is the lift of a simple arc of X/ι.

To see that every nonseparating curve fixed by ι is the core curve of a cylinder, let c be such a curve. As in the previous case, pick some $(X, f, \omega) \in \widetilde{\mathcal{H}}$ and a cylinder on it with core curve $f(d)$. Let γ and δ denote the (simple) arcs of Σ_g/ι corresponding to c and d, which connect the branch values of the associated cover $q : X \to X/\iota$.

We now recall that the hyperelliptic component $\mathcal{H} \subset \Omega\mathcal{M}(\underline{\kappa})$ is an orbifold $K(\pi, 1)$ for (an extension of) a surface braid group on X/ι [LM14, §1.4]. In particular, its geometric monodromy group $\mathcal{G}(\mathcal{H})$ contains a copy of the entire braid group B_q on the set of branch values of q which lift to regular points of (X, ω) (compare [Cal19, Proof of Corollary 2.6]). Since such a braid group acts transitively on the set of simple arcs connecting its points, we know there is an element of B_q taking δ to γ; hence by the Birman–Hilden theory (see Section 4.2 or [MW]) there is an element $g \in \mathcal{G}(\mathcal{H})$ taking d to c.

As above, the curve c is the core curve of a cylinder on

$$g \cdot (X, f, \omega) = (X, fg^{-1}, \omega),$$

and by Proposition 3.7, we have that (X, fg^{-1}, ω) is in $\widetilde{\mathcal{H}}$, finishing the proof. □

4. The sliding principle

The remainder of the paper is dedicated to the proof of Theorem B. As the proof will span several sections, we pause here to give an outline of the work remaining to be done (see also the outline given in Section 1).

4.1. **Outline of Theorem B.** At the highest level, the proof divides into two pieces: we first establish the "maximal" case $r = 2g - 2$ formulated in Theorem B.1 and B.2, and then we will use this to establish the case of general r as formulated in Theorem B.3.

The proof of the maximal case $r = 2g - 2$ divides further into two steps. The first step, carried out in Section 5, shows that the finite collection of twists described in Theorem B generate the full *admissible subgroup* $\mathcal{T}_\phi$ (c.f. Definition 2.6). The second step (Proposition 6.1) is to show that the admissible subgroup coincides with the spin structure stabilizer: $\mathcal{T}_\phi = \mathrm{Mod}_g[\phi]$. This is accomplished in Section 6 (more precisely, Sections 6.1–6.3). The work here applies to general r with essentially no modification, and in anticipation of the general case, we formulate and prove Proposition 6.1 for arbitrary r.

Given the maximal case, the proof in the general case is actually quite easy, and is handled in Section 6.5. In light of Proposition 6.1, it suffices to show that a finite collection of twists as given in Theorem B.3, together with the stabilizer of a lift of ϕ, generates the admissible subgroup $\mathcal{T}_\phi$.

Remark 4.1. Theorem B requires $g \geq 5$. This is necessary in only one place in the argument, Lemma 5.4. This lemma, which was proved in [Sal19], rests on the connectivity of a certain simplicial complex which is disconnected for $g < 5$. It is likely that Lemma 5.4 holds for $g \geq 3$, but to the best of the authors' knowledge, some substantial new ideas are needed to improve the range. Among other things, this would complete the classification of components of strata of marked abelian differentials in genera 3 and 4.

The remainder of the present Section 4 is devoted to establishing a versatile lemma known as the *sliding principle*. In the course of our later work in Section 5, we will often need to demonstrate that given a subgroup $\Gamma \leqslant \mathrm{Mod}(\Sigma_g)$ and two simple closed curves a and b, there is some $\gamma \in \Gamma$ such that $\gamma(a) = b$. The statements of the relevant lemmas (5.10, 5.11, and 5.13) are technical, and their proofs are necessarily computational. However, they are all manifestations of the sliding principle, which appears as Lemma 4.4 below as the culmination of a sequence of examples.

4.2. **Sliding along chains and Birman–Hilden theory.** The simplest example of the sliding principle is the braid relation: recall that if a and b are simple closed curves on a surface which intersect exactly once, then

$$T_a T_b T_a = T_b T_a T_b$$

and this element interchanges the curves a and b. More generally, if $(a_1, \ldots, a_n)$ is an n–chain of simple closed curves, then there is an element of $\Gamma := \langle T_{a_1}, \ldots, T_{a_n} \rangle$ which takes a_1 to a_n. We think of the curve a_1 as "sliding" along the chain $(a_1, \ldots, a_n)$ to a_n.

The theory of Birman and Hilden (see, e.g., [MW]) clarifies this phenomenon by identifying the group Γ as a braid group. This identification provides an explicit model for the action of Γ on simple closed curves, making the above statement apparent.

Namely, let W be the subsurface filled by $\{a_1, \ldots, a_n\}$; then W has a unique hyperelliptic involution ι which (setwise) fixes each a_i curve. The quotient $W/\langle \iota \rangle$ is a disk with $n+1$ marked points, and the Birman–Hilden theorem implies that

$$C_W(\iota) \cong B_{n+1} \tag{2}$$

where $C_W(\iota)$ is the centralizer of ι inside of $\mathrm{Mod}(W)$. The Dehn twist about a_i descends to the half–twist h_i interchanging the i^{th} and $(i+1)^{\text{st}}$ curves, and so we see that under the isomorphism (2), $\{T_{a_i}, \ldots, T_{a_n}\}$ corresponds to the standard Artin generators for B_{n+1}.

Now in B_{n+1} it is evident that any two half–twists h_i and h_j are conjugate, for example, by a braid which interchanges the i^{th} and $(i+1)^{\text{st}}$ strands with the j^{th} and $(j+1)^{\text{st}}$ strands. By the Birman–Hilden correspondence, T_{a_i} and T_{a_j} are conjugate in $C_W(\iota)$, and hence there is some element of Γ taking a_i to a_j (and vice–versa).

Similarly, any two sub–braid groups $B_{i,j} := \langle h_i, h_{i+1} \ldots, h_j\rangle$ and $B_{k,\ell} := \langle h_k, h_{k+1} \ldots, h_\ell\rangle$ generated by consecutive half–twists are conjugate in B_{n+1} if and only if $j - i = \ell - k$, that is, if they act on the same number of strands. In terms of subsurfaces, this means that if $Y_{i,j}$ and $Y_{k,\ell}$ denote the subsurfaces filled by the subchains $(a_i, \ldots, a_j)$ and $(a_k, \ldots, a_\ell)$, respectively, then there is some element $\gamma \in \Gamma$ which identifies the chains in an order–preserving way and hence takes $Y_{i,j}$ to $Y_{k,\ell}$.

The sliding principle for chains then boils down to using this action to transport curves living on $Y_{i,j}$ to curves on $Y_{k,\ell}$. In order to make this work, we need a coherent way of marking each subsurface.

By construction, $Y_{i,j} \setminus (a_i \cup \ldots \cup a_j)$ is a union of either one or two annuli, one for each component of $\partial Y_{i,j}$. In particular, the chain $(a_i, \ldots, a_j)$ determines a marking of $Y_{i,j}$ up to mapping classes of $Y_{i,j}$ preserving each curve of the chain. In the case at hand, the only such elements are Dehn twists about $\partial Y_{i,j}$ and the hyperelliptic involution.

Choose an orientation on a_1; by the transitivity of the Γ action on $\{a_i\}$, this specifies an orientation on each curve in the chain. Now the hyperelliptic involution reverses the orientation of each a_i, and hence the data of $(a_i, \ldots, a_j)$ together with their orientations is enough to determine a marking up to twists about $\partial Y_{i,j}$. Of course, the same procedure may be repeated for $Y_{k,\ell}$.

The identification $\gamma(Y_{i,j}) = Y_{k,\ell}$ should therefore be thought of as an identification of *marked* subsurfaces (up to twisting about $\partial Y_{i,j}$), and so can be used to transport any simple closed curve c supported on $Y_{i,j}$ to a curve $\gamma(c)$ supported on $Y_{k,\ell}$. Moreover, one can use the (signed) intersection pattern of c with the a_i to explicitly identify $\gamma(c)$ as a curve on $Y_{k,\ell}$.

Example 4.2. As a simple example of the sliding principle, consider the curves a_{2g} and a_{2g+2} shown in Figures 9 and 10 in Section 5 below. The curve a_{2g} is supported on the 5–chain $(a_4, \ldots, a_8)$, and a_{2g+2} is supported on $(a_2, \ldots, a_6)$. When $(a_4, \ldots, a_8)$ is slid to $(a_2, \ldots, a_6)$, this identification takes a_{2g} to a_{2g+2}.

Remark 4.3. A similar philosophy can be used to investigate the Γ action on curves which merely intersect W, but then one must be careful to take into account the incidence of the curve with ∂W and ensure that there is no twisting about $\partial Y_{i,j}$ (c.f. [Cal19, Lemmas A.4–7]).

4.3. **General sliding.** So far, what we have discussed is just an extended consequence of the Birman–Hilden correspondence for a hyperelliptic subsurface. The general sliding principle is a method for investigating the action on a union of such subsurfaces.

Let $\mathcal{C}$ be a set of simple closed curves on the surface Σ_g and set

$$\Gamma := \langle T_a : a \in \mathcal{C}\rangle.$$

Define the *intersection graph* $\Lambda_{\mathcal{C}}$ of $\mathcal{C}$ to have a vertex for each curve of $\mathcal{C}$, and two vertices to be connected by an edge if and only if the curves they represent intersect exactly once. Without loss of generality, we will assume that $\Lambda_{\mathcal{C}}$ is connected (otherwise each component can be dealt with separately).

Paths in the intersection graph $\Lambda_{\mathcal{C}}$ correspond to chains on the surface, which in turn fill hyperelliptic subsurfaces. By the discussion above, the Γ action can be used to slide curves supported in a neighborhood of $\mathcal{C}$ along paths in the intersection graph.

Generally, however, a curve cannot traverse all of $\Lambda_{\mathcal{C}}$ just by sliding. In particular, the subsurface carrying the curve can only transfer between chains or reverse the order of its filling chain when there is enough space for it to "turn around." For example, consider the set of curves $\mathcal{C} \subset \Sigma$ shown in Figure 5, whose intersection graph $\Lambda_{\mathcal{C}}$ is a tripod with legs of length 2, 2, and 6. We claim that Γ acts transitively on the set of (ordered) 3–chains in $\mathcal{C}$.

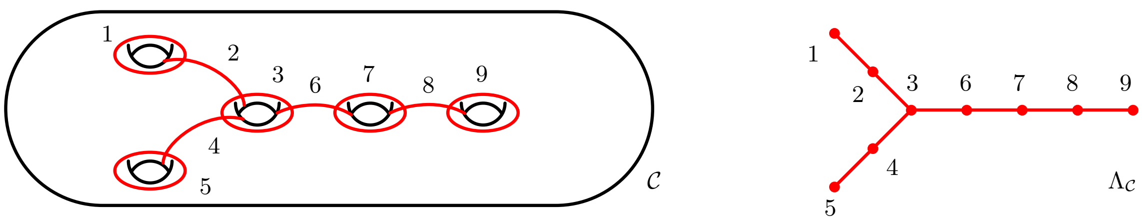


FIGURE 5. A set of simple closed curves $\mathcal{C}$ and its intersection graph $\Lambda_{\mathcal{C}}$. The Γ-action is transitive on 3–chains but not on 5–chains (at least not obviously so).

Indeed, given any 3–chain in $\mathcal{C}$, the sliding principle for chains implies that it can be taken to either (a_1, a_2, a_3) or (a_7, a_8, a_9), possibly with orientation reversed. The chains (a_1, a_2, a_3) and (a_7, a_8, a_9) are in turn related by sliding, so Γ acts transitively on the set of unordered 3–chains. Therefore, to see that Γ acts transitively on ordered 3–chains, it suffices to show that (a_1, a_2, a_3) is in the Γ orbit of (a_3, a_2, a_1). This follows by repeated sliding:

$$(a_1, a_2, a_3) \sim (a_3, a_4, a_5) \sim (a_9, a_8, a_7) \sim (a_3, a_2, a_1)$$

where we have written $c \sim c'$ to indicate that the chain c can be slid to the chain c' along a chain in $\mathcal{C}$.

However, Γ does not obviously act transitively on the set of 5–chains in $\mathcal{C}$. This can be explained by a lack of space in $\Lambda_{\mathcal{C}}$: the 5–chain $(a_1, \ldots, a_5)$ cannot be slid to lie entirely on one branch of $\Lambda_{\mathcal{C}}$, and so we cannot perform the same turning maneuvers as in the case of 3–chains.

We record this intuition in the following statement, the proof of which is just a repeated application of the sliding principle for chains.

Lemma 4.4 (The sliding principle)**.** *Suppose that $\mathcal{C}$ is a set of simple closed curves on a surface Σ_g and set*

$$\Gamma = \langle T_a : a \in \mathcal{C} \rangle.$$

Let $Y, W \subset \Sigma_g$ be subsurfaces filled by chains $(y_1, \ldots, y_k) \subset \mathcal{C}$ and $(w_1, \ldots, w_k) \subset \mathcal{C}$, respectively. If there exists a sequence $c_1, \ldots, c_n$ of chains in $\mathcal{C}$ such that

- *$(y_1, \dots, y_k)$ is a subchain of c_1*
- *$(w_1, \dots, w_k)$ is a subchain of c_n*
- *c_i and c_{i+1} overlap in a subchain of length at least k*

then there exists $\gamma \in \Gamma$ taking $(y_1, \dots, y_k)$ to $(w_1, \dots, w_k)$. Moreover, γ induces a natural identification of the simple closed curves supported entirely on Y with those supported entirely on W.

5. Finite generation of the admissible subgroup

With the preliminary sliding principle (Lemma 4.4) established, we begin the body of the proof of Theorem B. The first step is to show that the each of the finite collections of Dehn twists presented in Figures 1 and 2 generate their respective admissible subgroups $\mathcal{T}_\phi$. This is the main result of the next section.

Proposition 5.1. *In case 1 (respectively, case 2) of Theorem B, let Γ denote the group generated by the indicated collections of Dehn twists. Then $\Gamma = \mathcal{T}_\phi$, where ϕ is the $(2g-2)$–spin structure specified by assigning $\phi(c) = 0$ for every curve c appearing in Figure 1 (respectively, Figure 2).*

The proof of Proposition 5.1 is accomplished in stages. In Section 5.1 we recall the notion of a "spin subsurface push subgroup" $\widetilde{\Pi}(b)$ from [Sal19] and establish a criterion (Lemma 5.4) for Γ to contain $\mathcal{T}_\phi$ in terms of $\widetilde{\Pi}(b)$. In Section 5.2, we review the theory of "networks" from [Sal19], and use this to formulate an explicit generating set for $\widetilde{\Pi}(b)$ (Lemma 5.6). In Section 5.3, we briefly recall some relations in the mapping class group. Finally in Section 5.4 we show the containment $\widetilde{\Pi}(b) \leqslant \Gamma$, and so conclude the proof of Proposition 5.1.

As our ultimate goal is the proof of Proposition 5.1, throughout this section we consider only $(2g-2)$–spin structures, though most of the following statements hold for general r.

5.1. Spin subsurface push subgroups.

Here we recall the notion of a "spin subsurface push subgroup" from [Sal19, Section 8]. The main objective of this subsection is Lemma 5.4 below, which provides a criterion for a subgroup $H \leqslant \mathrm{Mod}(\Sigma_g)$ to contain the admissible subgroup $\mathcal{T}_\phi$ in terms of a spin subsurface push subgroup. Let Σ_g be a closed surface equipped with a $(2g-2)$–spin structure ϕ, and let $b \subset \Sigma_g$ be an essential, oriented, nonseparating curve satisfying $\phi(b) = -1$. Define S' to be the *closed* subsurface of Σ_g obtained by removing an open annular neighborhood of b; let Δ denote the boundary component of S' corresponding to the left side of b. Let $\overline{S'}$ denote the surface obtained from S' by capping off Δ by a disk.

Combining a suitable form of the Birman exact sequence (c.f. [FM11, Section 4.2.5]) with the inclusion homomorphism $i_* : \mathrm{Mod}(S') \to \mathrm{Mod}(\Sigma_g)$, the capping operation induces a homomorphism

$$\mathcal{P} : \pi_1(UT\overline{S'}) \to \mathrm{Mod}(\Sigma_g);$$

here $UT\overline{S'}$ denotes the unit tangent bundle to $\overline{S}'$. We call the image

$$\Pi(b) := \mathcal{P}(\pi_1(UT\overline{S'}))$$

a *subsurface push subgroup*[1] and remark that $\mathcal{P}$ can be shown to be an injection.

Definition 5.2 (Spin subsurface push subgroup)**.** Let Σ_g be a closed surface equipped with a $(2g-2)$–spin structure ϕ, and let $b \subset \Sigma_g$ be an essential, oriented, nonseparating curve satisfying $\phi(b) = -1$. The *spin subsurface push subgroup* $\widetilde{\Pi}(b)$ is[2] the intersection

$$\widetilde{\Pi}(b) := \Pi(b) \cap \operatorname{Mod}_g[\phi].$$

Lemma 5.3 ([Sal19], Lemma 8.1)**.** *The spin subsurface push subgroup* $\widetilde{\Pi}(b)$ *is a finite–index subgroup of* $\Pi(b)$*. It is characterized by the group extension*

$$1 \to \langle T_b^{2g-2} \rangle \to \widetilde{\Pi}(b) \to \pi_1(\overline{S'}) \to 1; \tag{3}$$

the map $\widetilde{\Pi}(b) \to \pi_1(\overline{S'})$ *is induced by the capping map* $S' \to \overline{S'}$ *where the boundary component corresponding to the left side of* b *is capped off with a punctured disk.*

The following Lemma 5.4 was established in [Sal19]. It shows that a spin subsurface push subgroup $\widetilde{\Pi}(b)$ is "not far" from containing the entire admissible subgroup $\mathcal{T}_\phi$. In the next subsection, we will make this more concrete by finding an explicit finite set of generators for $\widetilde{\Pi}(b)$, and in Section 5.4 we will do the work necessary to show that Γ contains this generating set, and consequently to show the equality $\Gamma = \mathcal{T}_\phi$.

Lemma 5.4 (C.f. [Sal19], Lemma 8.2)**.** *Let* ϕ *be a* $(2g-2)$*–spin structure on a closed surface* Σ_g *for* $g \geq 5$*. Let* (a, a', b) *be an ordered* 3*-chain of curves with* $\phi(a) = \phi(a) = 0$ *and* $\phi(b) = -1$*. Let* $H \leqslant \operatorname{Mod}(\Sigma_g)$ *be a subgroup containing* $T_a, T_{a'}$ *and the spin subsurface push group* $\widetilde{\Pi}(b)$*. Then* H *contains* $\mathcal{T}_\phi$*.*

5.2. **Networks.** In this subsection we describe an explicit finite generating set for $\widetilde{\Pi}(b)$, stated as Lemma 5.6. This is formulated in the language of "networks" from [Sal19, Section 9].

Definition 5.5 (Networks)**.** Let $S = \Sigma_{g,b}^n$ be a surface, viewed as a compact surface with marked points. A *network* on S is any collection $\mathcal{N} = \{a_1, \ldots, a_n\}$ of simple closed curves on S, disjoint from any marked points, such that $\#(a_i \cap a_j) \leqslant 1$ for all pairs of curves $a_i, a_j \in \mathcal{N}$, and such that there are no triple intersections. A network $\mathcal{N}$ has an associated *intersection graph* $\Lambda_\mathcal{N}$, whose vertices correspond to curves $x \in \mathcal{N}$, with vertices x, y adjacent if and only if $\#(x \cap y) = 1$. A network is said to be *connected* if $\Lambda_\mathcal{N}$ is connected, and *arboreal* if $\Lambda_\mathcal{N}$ is a tree. A network is *filling* if

$$S \setminus \bigcup_{a \in \mathcal{N}} a$$

is a disjoint union of disks and boundary-parallel annuli; each disk component is allowed to contain at most one marked point of S and each annulus component may not contain any.

[1] We have attempted to improve the notation introduced in [Sal19, Section 8] where the corresponding subsurface push subgroup was denoted $\Pi(S', \Delta)$.

[2] Again, the notation here differs slightly with [Sal19], where the spin subsurface push subgroup is denoted $\widetilde{\Pi}(\Sigma_g \setminus \{b\})$.

The following lemma provides the promised explicit finite generating set for (a supergroup of) $\widetilde{\Pi}(b)$. As always, we assume that ϕ is a $(2g-2)$–spin structure on Σ_g with $g \geq 5$. Let $b \subset \Sigma_g$ be an essential, oriented, nonseparating curve satisfying $\phi(b) = -1$, and consider the surface $\overline{S'}$ of Section 5.1 as well as the spin subsurface push subgroup $\widetilde{\Pi}(b)$.

Lemma 5.6. *Suppose $\mathcal{N}$ is an arboreal filling network on $\overline{S'}$, and suppose that there exist $a, a' \in \mathcal{N}$ such that $a \cup a' \cup b$ forms a pair of pants on Σ_g. Let $H \leqslant \mathrm{Mod}(\Sigma_g)$ be a subgroup containing T_a for each $a \in \mathcal{N}$ and T_b^{2g-2}. Then $\widetilde{\Pi}(b) \leqslant H$.*

Proof. This is an amalgamation of the results of [Sal19, Section 9]. See especially the proof of [Sal19, Theorem 9.5] as well as [Sal19, Lemma 9.4]. In the latter, we have replaced the hypothesis $P(a_1) \in H$ by the requirement that $a \cup a' \cup b$ form a pair of pants; in this case, the corresponding push map is given simply by $T_a T_{a'}^{-1} \in H$. □

5.3. **Relations in the mapping class group.** In preparation for the explicit computations to be carried out in Section 5.4, we collect here some relations within the mapping class group. The chain and lantern relations are classical; a discussion of the D relation can be found in, e.g., [Sal19, Section 2.3].

Lemma 5.7 (The chain relation)**.** *Let $(a_1, \ldots, a_k)$ be a chain of simple closed curves. If k is even, let d denote the single boundary component of the subsurface determined by $(a_1, \ldots, a_k)$, and if k is odd, let d_1, d_2 denote the two boundary components.*

- *If k is even, then $T_d = (T_{a_1} \ldots T_{a_k})^{2k+2}$.*
- *If k is odd, then $T_{d_1} T_{d_2} = (T_{a_1} \ldots T_{a_k})^{k+1}$.*

Lemma 5.8 (The lantern relation)**.** *Let a, b, c, d, x, y, z be the simple closed curves shown in Figure 6. Then*

$$T_a T_b T_c T_d = T_x T_y T_z.$$

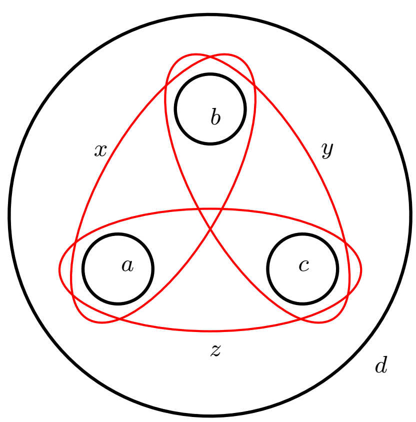


FIGURE 6. The lantern relation.

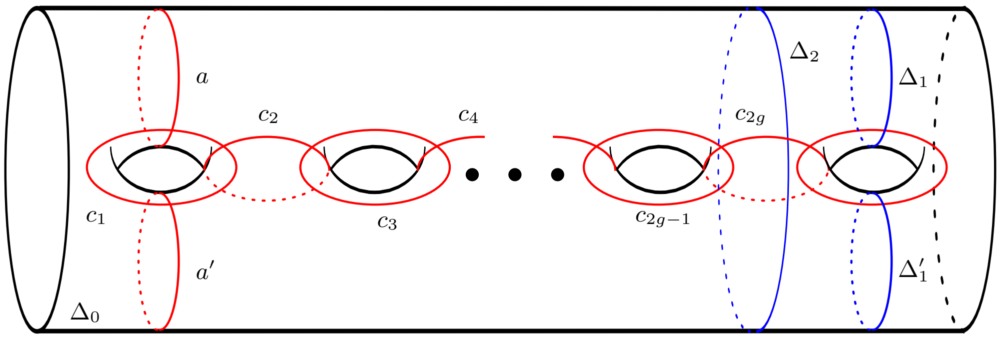


FIGURE 7. The configuration of curves used in the D relation.

Lemma 5.9 (The D relation)**.** *Let $n \geq 3$ be given, and express $n = 2g + 1$ or $n = 2g + 2$ according to whether n is odd or even. With reference to Figure 7, let H_n be the group generated by elements of the form T_x, with $x \in \mathscr{D}_n$ one of the curves below:*

$$\mathscr{D}_n = \{a, a', c_1, \ldots, c_{n-2}\}.$$

Then for $n = 2g + 1$ odd,

$$T_{\Delta_0}^{2g-1} T_{\Delta_2} \in H_n,$$

and for $n = 2g + 2$ even,

$$T_{\Delta_0}^{g} T_{\Delta_1} T_{\Delta_1'} \in H_n.$$

5.4. **Generating the spin subsurface push subgroup.** In this section we complete the proof of Proposition 5.1. Appealing to Lemma 5.4, we must find suitable curves a, a', b such that $T_a, T_{a'}, \widetilde{\Pi}(b) \in \Gamma$. Following Lemma 5.6, it suffices to find curves a, a' and a suitable arboreal filling network $\mathcal{N}$ on $\overline{S'}$. Recall that Proposition 5.1 treats two cases, corresponding to the two generating sets appearing in cases 1, 2 of Theorem B. The arguments for cases 1 and 2 are different, and are completed in Lemmas 5.10 and 5.13, respectively.

Case 1. With reference to Figure 8, observe that the 3–chain (a_4, a_3, b) satisfies the hypotheses of the 3–chain (a, a', b) of Lemma 5.4. It therefore suffices to show that $\Gamma = \langle T_{a_i} \mid 0 \leqslant i \leqslant 2g-1 \rangle$ contains $\widetilde{\Pi}(b)$. Observe that $a_0 \cup a_4 \cup b$ forms a pair of pants and that $\{T_{a_i} \mid 0 \leqslant i \leqslant 2g-1,\ i \neq 3\}$ forms an arboreal filling network on $\overline{S'}$. According to Lemma 5.6, it therefore suffices to show that $T_b^{2g-2} \in \Gamma$.

Lemma 5.10. *Let b be the curve indicated in Figure 8. Then $T_b^{2g-2} \in \Gamma$ in case 1 of Theorem B. Consequently, $\Gamma = \mathcal{T}_\phi$ in case 1.*

Proof. We will first show that $T_c^{2g-2} \in \Gamma$ for the curve c shown in Figure 8, and then we will conclude the argument by showing that b and c are in the same orbit of the Γ–action on simple closed curves. Consider the $\mathscr{D}_{2g-3}$–configuration determined by the curves $a_0, a_2, a_5, a_6, \ldots, a_{2g-1}$ with boundary components c, c'. Applying the D relation (Lemma 5.9),

$$T_c^{2g-5} T_{c'} \in \Gamma.$$

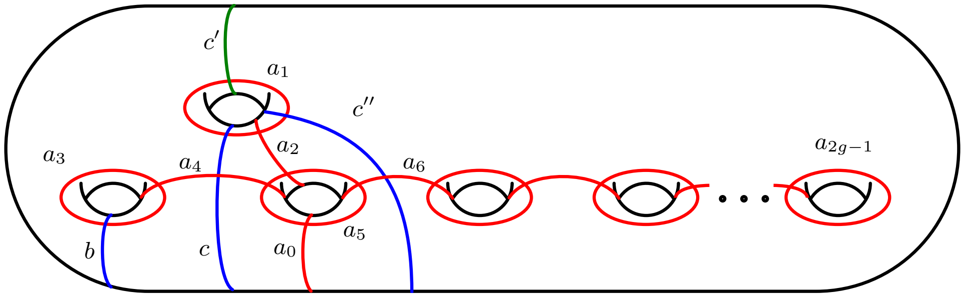


FIGURE 8. The configuration of curves defining Γ in case 1, along with the auxiliary curves b, c, c', c''. For clarity, the portions of the curves on the back side have been omitted; in all cases, they continue on the back as the mirror image (relative to the plane of the page).

Next, consider the $\mathscr{D}_5$–configuration determined by a_0, a_2, a_3, a_4, a_5 with boundary components c', c''. Applying the D relation to this, we find

$$T_{c'} T_{c''}^3 \in \Gamma.$$

Finally, the chain relation as applied to (a_0, a_5, a_2) shows that

$$T_c T_{c''} \in \Gamma;$$

combining these three results shows $T_c^{2g-2} \in \Gamma$.

The curves b, c are boundary components of the 3–chains (a_4, a_5, a_0) and (a_2, a_5, a_0), respectively. Since we can slide the 3–chains to each other via

$$(a_4, a_5, a_0) \sim (a_3, a_4, a_5) \sim (a_5, a_6, a_7) \sim (a_1, a_2, a_5) \sim (a_2, a_5, a_0)$$

the sliding principle (Lemma 4.4) shows that b can be taken to c by an element of Γ. □

Case 2. In Case 2, the arboreal network we use does not consist entirely of the curves $a_0, \dots, a_{2g-1}$, and so our first item of business is to see that Γ contains the admissible twists $T_{a_{2g}}, T_{a_{2g+1}}$ shown in Figure 9. In Lemma 5.13, we will also need to use the twist $T_{a_{2g+2}}$ for the curve a_{2g+2} shown in Figure 10, and in fact we will obtain $T_{a_{2g}}, T_{a_{2g+1}} \in \Gamma$ from the containment $T_{a_{2g+2}} \in \Gamma$ established in Lemma 5.11.

Lemma 5.11. *In case 2 of Theorem B, we have $T_{a_{2g+2}} \in \Gamma$ for the curve a_{2g+2} shown in Figure 10.*

Proof. This is closely related to the sliding principle. One verifies (see Figure 11) that

$$(T_{a_5} T_{a_4} T_{a_3} T_{a_2})(T_{a_6} T_{a_5} T_{a_4} T_{a_3})(T_{a_7} T_{a_6} T_{a_5} T_{a_4})(T_{a_0} T_{a_5} T_{a_6} T_{a_7})(a_{2g+2}) = a_0.$$

This product of twists is an element of Γ, showing that $T_{a_{2g+2}}$ is conjugate to T_{a_0} by an element of Γ, and hence $T_{a_{2g+2}} \in \Gamma$ itself. □

Lemma 5.12. *The admissible twists $T_{a_{2g}}$ and $T_{a_{2g+1}}$ shown in Figure 9 are both contained in Γ.*

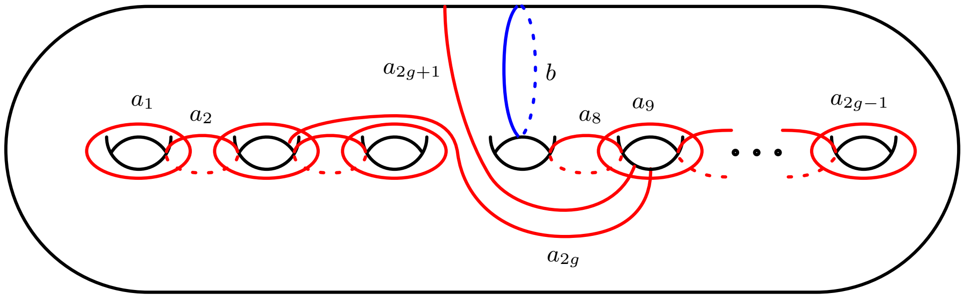


FIGURE 9. The filling arboreal network $\mathcal{N}$ used in Case 2. Curves a_{2g}, a_{2g+1} continue on the back of the surface as the mirror image.

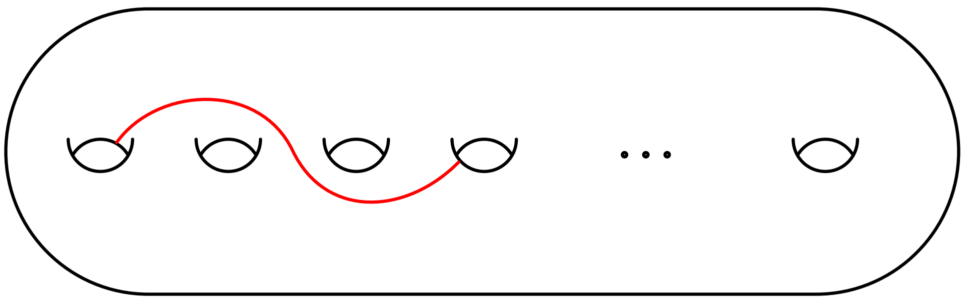


FIGURE 10. The curve a_{2g+2} used in the proof of Lemma 5.13. It continues on the back side as the mirror image.

Proof. The curve a_{2g} is obtained from a_{2g+2} by sliding (see Example 4.2). To find a sequence of twists about elements $T_{a_i} (0 \leqslant i \leqslant 2g-1)$ taking a_{2g} to a_{2g+1}, we observe that the sequence of slides

$$(a_4, \ldots, a_8) \sim (a_5, \ldots, a_9) \sim (a_0, a_5, \ldots, a_8)$$

takes the curve a_{2g} to a_{2g+1}. □

Lemma 5.13. *Let b be the curve indicated in Figure 12. Then $T_b^{2g-2} \in \Gamma$ in case 2 of Theorem B. Consequently, $\Gamma = \mathcal{T}_\phi$ in case 2.*

Proof. As in Case 1, we will first show $T_{b'}^{2g-2} \in \Gamma$ for a different curve b', and subsequently show that b, b' are in the same Γ–orbit. Consider first the $\mathscr{D}_5$–configuration determined by a_0, a_4, a_5, a_6, a_7 with boundary components b', c. By the D relation (Lemma 5.9)

$$T_{b'}^3 T_c \in \Gamma.$$

Next consider the $\mathscr{D}_{2g-3}$–configuration determined by $a_0, a_4, a_5, \ldots, a_{2g-1}$. By the D relation,

$$T_{b'}^{2g-5} T_{c'} \in \Gamma.$$

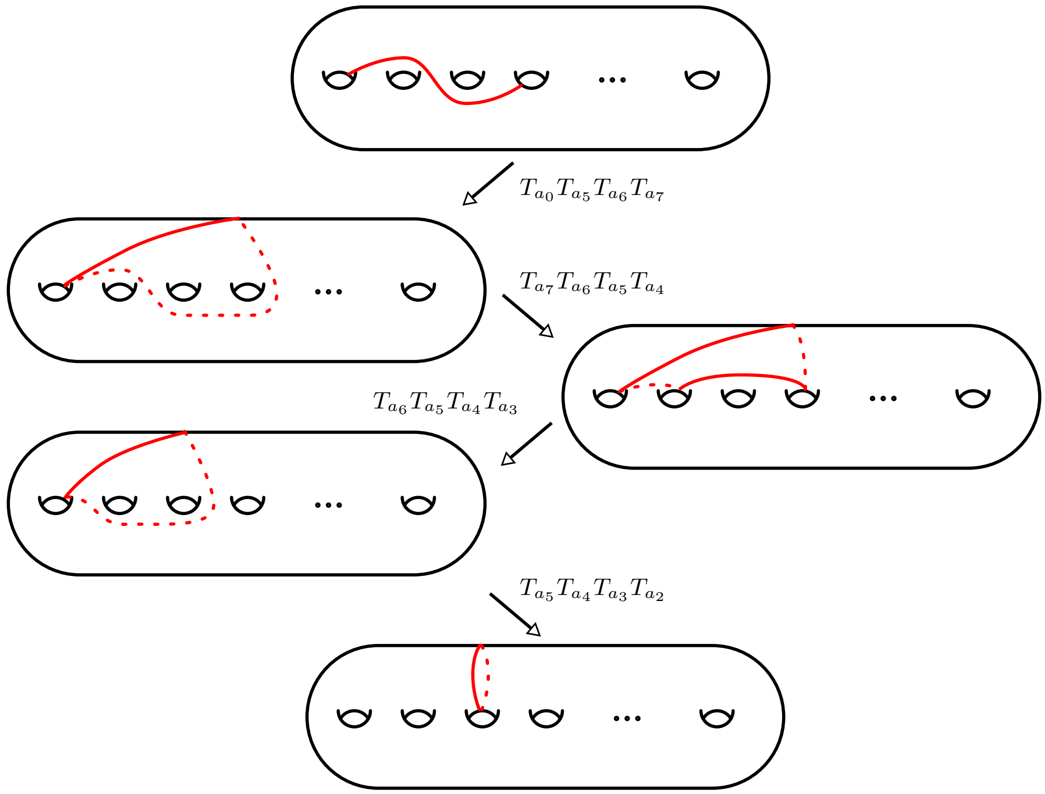


FIGURE 11. The sequence of twists used to take a_{2g+2} to a_0 in Lemma 5.11.

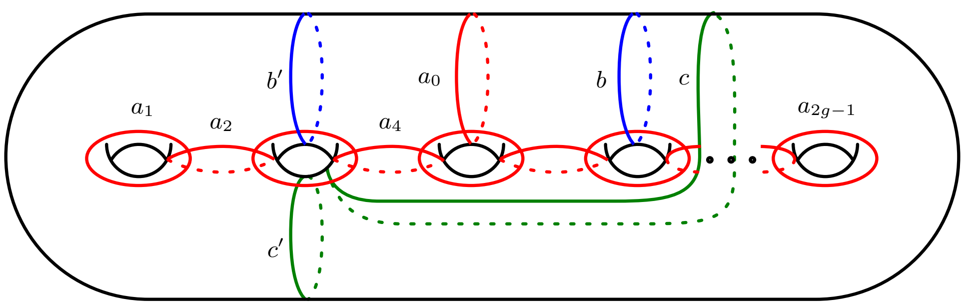


FIGURE 12. The configuration of curves defining Γ in case 2, along with the auxiliary curves b', c, c'.

As in Lemma 5.10, it now suffices to show that $T_c T_{c'} \in \Gamma$. To see this, observe that the sequence $(a_0, a_5, a_6, a_7, a_{2g+2}, a_1, a_2)$ forms a 7–chain with boundary components c, c'. Thus

$$T_c T_{c'} \in \Gamma$$

by an application of the chain relation. Combining these elements yields $T_{b'}^{2g-2} \in \Gamma$; the Γ–equivalence of b and b' follows by the sliding principle (Lemma 4.4).

At this point, we have shown that $T_a \in \Gamma$ for each element a of the network $\mathcal{N}$ defined in Figure 9, and we have shown that $T_b^{2g-2} \in \Gamma$. To complete the proof of Lemma 5.13, it remains only to appeal to Lemma 5.4 to see that these elements generate $\mathcal{T}_\phi$. The chain (a_8, a_7, b) satisfies the hypothesis of Lemma 5.4, and so it remains to show that $\widetilde{\Pi}(b) \leqslant \Gamma$. The network $\mathcal{N}$ is evidently arboreal and fills $\overline{S'}$, and $a_8 \cup a_{2g+1} \cup b$ forms a pair of pants. Combining Lemma 5.12 with the result $T_b^{2g-2} \in \Gamma$ shows that the hypotheses of Lemma 5.6 are satisfied, completing the argument. □

6. Spin structure stabilizers and the admissible subgroup

The second step in the proof of Theorem B is to show that the admissible subgroup coincides with the full spin structure stabilizer. This is the counterpart to [Sal19, Propositions 5.1 and 6.2]. Those results only applied for g sufficiently large[3] and imposed the requirement $r < g - 1$. Moreover, in the case of r even, [Sal19, Proposition 6.2] does not assert the equality $\mathcal{T}_\phi = \mathrm{Mod}_g[\phi]$, only that $[\mathrm{Mod}_g[\phi] : \mathcal{T}_\phi] < \infty$. Proposition 6.1 deals with all of these issues at once.

Proposition 6.1. *Let ϕ be an r–spin structure on a surface Σ_g of genus $g \geq 3$. Then $\mathcal{T}_\phi = \mathrm{Mod}_g[\phi]$.*

When $r = 2g - 2$, this completes the proof of Theorem B1 and 2. The proof of Theorem B3 follows quickly, and is contained in Section 6.5.

The proof of the Proposition is again accomplished in stages. In Section 6.1 we outline the strategy and establish the first of three substeps. In Section 6.2, we discuss various versions of the "change–of–coordinates principle" in the presence of an r–spin structure. In the following Sections 6.3 and 6.4 we use these results to carry out the second and third substeps, respectively.

6.1. Outline – the Johnson filtration.

Once again the outline follows that given in [Sal19] – compare to Sections 5 and 6 therein. For any r–spin structure ϕ, there is the evident containment

$$\mathcal{T}_\phi \leqslant \mathrm{Mod}_g[\phi].$$

To obtain the opposite containment, we appeal to the *Johnson filtration* of $\mathrm{Mod}(\Sigma_g)$. For our purposes, we need only consider the three–step filtration

$$\mathcal{K}_g \leqslant \mathcal{I}_g \leqslant \mathrm{Mod}(\Sigma_g).$$

The subgroup $\mathcal{I}_g$ is the *Torelli group*. It is defined as the kernel of the symplectic representation $\Psi : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ which sends a mapping class f to its induced action f_* on $H_1(\Sigma_g; \mathbb{Z})$. Set

$$H := H_1(\Sigma_g; \mathbb{Z}).$$

The group $\mathcal{K}_g$ is the *Johnson kernel*. It is defined as the kernel of the *Johnson homomorphism* (see Lemma 6.10)

$$\tau : \mathcal{I}_g \to \wedge^3 H/H.$$

There is an alternate characterization of $\mathcal{K}_g$ due to Johnson.

[3] There is a typo in the statement of [Sal19, Proposition 5.1] – the range should be $g \geq 5$, not $g \geq 3$ as claimed.

Theorem 6.2 (Johnson [Joh85])**.** *Let $\mathcal{C}$ denote the set of separating curves $c \subset \Sigma_g$ where c bounds a subsurface of genus at most 2. For $g \geq 3$, there is an equality*

$$\mathcal{K}_g = \langle T_c \mid c \in \mathcal{C} \rangle.$$

The containment $\mathrm{Mod}_g[\phi] \leqslant \mathcal{T}_\phi$ will follow from a sequence of three lemmas. In preparation for Lemma 6.3, recall from Section 2.3 that an r–spin structure for r even determines an associated quadratic form (Remark 2.11), as well as the algebraic stabilizer subgroup $\mathrm{Sp}(2g, \mathbb{Z})[q]$ of Definition 2.14.

Lemma 6.3 (Step 1)**.** *Fix $g \geq 3$ and let ϕ be an r–spin structure on Σ_g. If r is odd, there is an equality*

$$\Psi(\mathrm{Mod}_g[\phi]) = \Psi(\mathcal{T}_\phi) = \mathrm{Sp}(2g, \mathbb{Z}).$$

If r is even, let q denote the quadratic form on $H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ associated to ϕ. Then there is an equality

$$\Psi(\mathrm{Mod}_g[\phi]) = \Psi(\mathcal{T}_\phi) = \mathrm{Sp}(2g, \mathbb{Z})[q].$$

Lemma 6.4 (Step 2)**.** *For $g \geq 3$, both $\mathrm{Mod}_g[\phi]$ and $\mathcal{T}_\phi$ contain $\mathcal{K}_g$.*

Lemma 6.5 (Step 3)**.** *For $g \geq 3$ there is an equality*

$$\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g) = \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$$

of subgroups of $\wedge^3 H/H$.

Lemma 6.3 was established as [Sal19, Lemmas 5.4 and 6.4]. Lemma 6.4 is established in Section 6.3. The proof of Lemma 6.5 relies on the previous two steps, and is established in Section 6.4.

Before proceeding with the argument, we pause to complete our proof of Corollary 1.3.

Proof of Corollary 1.3. The monodromy $\overline{\rho} : \pi_1^{\mathrm{orb}}(\mathcal{H}, (X, \omega)) \to \mathrm{Sp}(2g; \mathbb{Z})$ of the bundle $H_1(\mathcal{H})$ factors through the geometric monodromy $\rho : \pi_1^{\mathrm{orb}}(\mathcal{H}, (X, \omega)) \to \mathrm{Mod}_g[\phi]$:

$$\overline{\rho} = \Psi \circ \rho.$$

From (1) in the proof of Theorem A, ρ surjects onto the spin structure stabilizer $\mathrm{Mod}_g[\phi]$. The result now follows from Lemma 6.3. □

6.2. **Change–of–coordinates.** The classical change–of–coordinates principle (c.f. [FM11, Section 1.3]) describes the orbits of various configurations of curves and subsurfaces under the action of the mapping class group. When the underlying surface is equipped with an r–spin structure ϕ, we will need to understand $\mathrm{Mod}_g[\phi]$–orbits of configurations as well. The results below (Lemma 6.6–6.9) all present various facets of the change–of–coordinates principle in the presence of a spin structure. We will not prove these statements; Lemmas 6.6 and 6.7 are taken from [Sal19, Section 4] verbatim, while Lemmas 6.8 and 6.9 follow easily from the techniques therein.

Lemma 6.6. *Let r be an odd integer, and let Σ_g be a surface of genus $g \geq 2$ equipped with an r–spin structure ϕ. Let $S \subset \Sigma_g$ be a subsurface of genus $h \geq 2$ with a single boundary component. Then the following assertions hold:*

(1) *For any* $2h$*–tuple* $(i_1, j_1, \ldots, i_h, j_h)$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some geometric symplectic basis* $\mathcal{B} = \{a_1, b_1, \ldots, a_h, b_h\}$ *for* S *with* $\phi(a_\ell) = i_\ell$ *and* $\phi(b_\ell) = j_\ell$ *for all* $1 \leqslant \ell \leqslant h$*,*
(2) *For any* $2h$*–tuple* $(k_1, \ldots, k_{2h})$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some chain* $(a_1, \ldots, a_{2h})$ *of curves on* S *such that* $\phi(a_\ell) = k_\ell$ *for all* $1 \leqslant \ell \leqslant 2h$*.*

Lemma 6.7. *Let* r *be an even integer, and let* Σ_g *be a surface of genus* $g \geq 2$ *equipped with an* r*–spin structure* ϕ*. Let* $S \subset \Sigma_g$ *be a subsurface of genus* $h \geq 2$ *with a single boundary component. Then the following assertions hold:*

(1) *For a given* $2h$*–tuple* $(i_1, j_1, \ldots, i_h, j_h)$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some geometric symplectic basis* $\mathcal{B} = \{a_1, b_1, \ldots, a_h, b_h\}$ *for* S *with* $\phi(a_\ell) = i_\ell$ *and* $\phi(b_\ell) = j_\ell$ *for* $1 \leqslant \ell \leqslant h$ *if and only if the parity of the spin structure defined by these conditions agrees with the parity of the restriction* $\phi|_S$ *to* S*.*
(2) *For* any $(2h-2)$*–tuple* $(i_1, j_1, \ldots, i_{h-1}, j_{h-1})$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some geometric symplectic basis* $\mathcal{B} = \{a_1, b_1, \ldots, a_h, b_h\}$ *for* S *with* $\phi(a_\ell) = i_\ell$ *and* $\phi(b_\ell) = j_\ell$ *for* $1 \leqslant \ell \leqslant h-1$*.*
(3) *For a given* $2h$*–tuple* $(k_1, \ldots, k_{2h})$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some chain* $(a_1, \ldots, a_{2h})$ *of curves on* S *such that* $\phi(a_\ell) = k_\ell$ *for all* $1 \leqslant \ell \leqslant 2h$ *if and only if the parity of the spin structure defined by these conditions agrees with the parity of the restriction* $\phi|_S$ *to* S*.*
(4) *For* any $(2h-2)$*–tuple* $(k_1, \ldots, k_{2h-2})$ *of elements of* $\mathbb{Z}/r\mathbb{Z}$*, there is some chain* $(a_1, \ldots, a_{2h-2})$ *of curves on* S *such that* $\phi(a_\ell) = k_\ell$ *for all* $1 \leqslant \ell \leqslant 2h-2$*.*

One of the most important iterations of the change–of–coordinates principle is that the spin stabilizer subgroup acts transitively on the set of all curves with a given winding number.

Lemma 6.8. *Let* ϕ *be a* r*–spin structure, and let* $c, d \subset \Sigma_g$ *be nonseparating curves. If* $\phi(c) = \phi(d)$*, then there is some* $f \in \mathrm{Mod}_g[\phi]$ *such that* $f(c) = d$*.*

One can also use this principle to find curves in a given homology class with given winding number (subject to Arf invariant restrictions, when applicable).

Lemma 6.9. *Let* ϕ *be an* r*–spin structure on* Σ_g *and let* $z \in H_1(\Sigma_g; \mathbb{Z})$ *be fixed. If* r *is odd, then for any element* $k \in \mathbb{Z}/r\mathbb{Z}$*, there is a simple closed curve* c *satisfying* $\phi(c) = k$ *and* $[c] = z$*. If* r *is even, let* $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ *denote the mod* 2 *value of* $\phi(z)$ *in the sense of Lemma 2.9. Then there exists a simple closed curve* c *satisfying* $\phi(c) = k$ *and* $[c] = z$ *if and only if* $k \equiv \varepsilon \pmod 2$*.*

6.3. **Step 2: Containment of the Johnson kernel.** Our objective in this section is to establish Lemma 6.4, showing $\mathcal{K}_g < \mathcal{T}_\phi$.

Proof. Suppose that ϕ is an r–spin structure, and $\widetilde{\phi}$ is a $(2g-2)$–spin structure which refines ϕ. Then $\mathcal{T}_{\widetilde{\phi}} \leqslant \mathcal{T}_\phi$, and hence it suffices to prove the lemma in the case when ϕ is a $(2g-2)$–spin structure.

By Theorem 6.2, it suffices to prove that if c is a separating curve bounding a subsurface of genus at most 2, then $T_c \in \mathcal{T}_\phi$.

Genus 1. We begin by considering the genus 1 case, so suppose that a is a curve which bounds a genus 1 subsurface W_a. Observe that if either

- $g \equiv 2, 3 \pmod 4$ and $\text{Arf}(\phi) = 1 + \text{Arf}(\phi|_{W_a})$ or
- $g \equiv 0, 1 \pmod 4$ and $\text{Arf}(\phi) = \text{Arf}(\phi|_{W_a})$,

then the complementary subsurface $\Sigma_g \setminus W_a$ has

- $g(\Sigma_g \setminus W_a) \equiv 1, 2 \pmod 4$ and $\text{Arf}\left(\phi|_{\Sigma_g \setminus W_a}\right) = 1$ or
- $g(\Sigma_g \setminus W_a) \equiv 3, 0 \pmod 4$ and $\text{Arf}\left(\phi|_{\Sigma_g \setminus W_a}\right) = 0$,

respectively. In either of the above cases, there exists a maximal chain of admissible curves on $\Sigma_g \setminus W_a$ by the change–of–coordinates principle (Lemma 6.7.3), and hence by the chain relation (Lemma 5.7), $T_a \in \mathcal{T}_\phi$.

Suppose now that we are not in one of the cases above, so the complementary subsurface $\Sigma_g \setminus W_a$ does not admit a maximal chain of admissible curves. In order to exhibit the twist on a, we will form a lantern relation and prove that the other terms in the relation lie in $\mathcal{T}_\phi$.

Take some curve b on $\Sigma_g \setminus W_a$ bounding a subsurface W_b of genus 1 whose complement admits a maximal admissible chain (such a curve may be found by properly specifying the ϕ values on a pair of dual elements in a geometric symplectic basis and then taking a neighborhood of their union). The chain relation (Lemma 5.7) then implies that $T_b \in \mathcal{T}_\phi$.

Let c be any curve in $\Sigma_g \setminus (W_a \cup W_b)$ such that $\phi(c) = -2$, and take d to be a curve which together with a, b, and c bounds a four–holed sphere (in the case $g = 3$, necessarily $c = d$, but this is not a problem). By homological coherence, $\phi(d) = -2$. These curves fit into a lantern relation as shown in Figure 13.

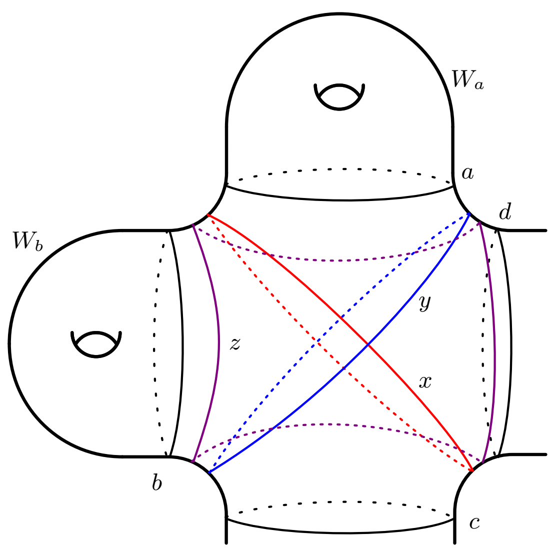


Figure 13. The lantern for genus 1.

By construction, $c \cup d$ bounds a subsurface $V \cong \Sigma_2^2$. Choose a subsurface $V' \subset V$ with a single boundary component y such that V' contains both W_a and W_b. Since the Arf invariants of W_a and W_b are of opposite parity, we have

$$\text{Arf}\,(\phi|_{V'}) = 1.$$

By the change of coordinates principle (Lemma 6.7.3) there exists a maximal chain $\{a_1, a_2, a_3, a_4\}$ of admissible curves on V'; by the chain relation, $T_y \in \mathcal{T}_\phi$. Now let $a_5 \subset V$ be a curve disjoint from a_2 so that $i(a_4, a_5) = 1$ and which together with a_1, a_3, and c bounds a four–holed sphere. By homological coherence (Lemma 2.3), we have that $\phi(a_5) = 0$. Therefore, $\{a_1, \ldots, a_5\}$ is a maximal chain of admissible curves on V and so by the chain relation, we have that $T_c T_d \in \mathcal{T}_\phi$.

Finally, we note that the pairs (c, x) and (d, z) each bound subsurfaces of genus 1 with two boundary components. Since $\phi(c) = \phi(d) = -2$, homological coherence implies that x and z must both be admissible.

Applying the lantern relation (Lemma 5.8), we have that

$$T_a = (T_b T_c T_d)^{-1}(T_x T_y T_z) \in \mathcal{T}_\phi.$$

Observe that if $g = 3$, then this is enough to finish the proof, since every separating twist is of genus 1.

Genus 2. Now suppose that $g \geq 4$ and let x be a curve bounding a subsurface W_x of genus 2 (this choice of label will allow Figures 13 and 14 to share a labeling system). Observe that if the Arf invariant of $\phi|_{W_x}$ is odd, then by the change–of–coordinates principle (Lemma 6.7.3), W_x admits a maximal chain of admissible curves and so by applying the chain relation, $T_x \in \mathcal{T}_\phi$.

So suppose that $\mathrm{Arf}\,(\phi|_{W_x}) = 0$. By the change–of–coordinates principle (in particular, Lemma 6.7.1), we can choose two disjoint subsurfaces $W_b \subset W_x$ and $W_c \subset W_x$ each homeomorphic to Σ_1^1 such that

$$\mathrm{Arf}\,(\phi|_{W_b}) = \mathrm{Arf}\,(\phi|_{W_c}) = 1.$$

Let their corresponding boundaries be b and c. Again appealing to Lemma 6.7.1, choose a to be a curve bounding a subsurface W_a of $\Sigma_g \setminus W_x$ with $W_a \cong \Sigma_1^1$ such that

$$\mathrm{Arf}\,(\phi|_{W_a}) = 0.$$

By the genus–1 case established above, we know that $T_a, T_b, T_c \in \mathcal{T}_\phi$. Finally, choose d to be any curve in $\Sigma_g \setminus (W_x \cup W_a)$ which bounds a pair of pants together with a and x. The curves then fit into a lantern relation as in Figure 14.

Let W_d denote the subsurface bounded by d which contains W_a, W_b, W_c. By construction, we have $\mathrm{Arf}(\phi|_{W_d}) = 0$. By the change of coordinates principle (Lemma 6.7.3), it follows that W_d admits a maximal chain of admissible curves, and so $T_d \in \mathcal{T}_\phi$ by the chain relation.

Finally, observe that the curves y and z shown in Figure 14 bound subsurfaces of genus 2 with odd Arf invariant, and so both admit maximal chains of admissible curves. Thus $T_x, T_y \in \mathcal{T}_\phi$.

Again applying the lantern relation (Lemma 5.8), we see

$$T_x = (T_a T_b T_c T_d)(T_y T_z)^{-1} \in \mathcal{T}_\phi.$$

Therefore, since the separating twists of genus one and two generate the Johnson kernel (Theorem 6.2), we see that $\mathcal{K}_g < \mathcal{T}_\phi$. □

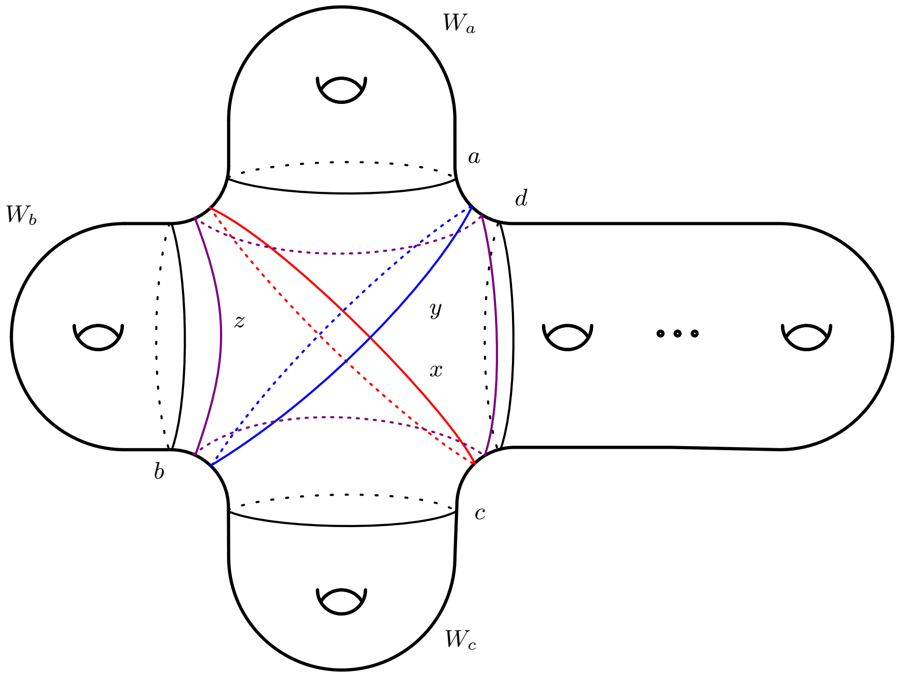


FIGURE 14. The lantern for genus 2 separating twists.

6.4. **Step 3: Intersection with the Torelli group.** In order to show the equality $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g) = \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$, it is necessary to give a precise description of the subgroup $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g)$. We begin with a brief summary of the theory of the Johnson homomorphism. It is not necessary to know a construction; we content ourselves with a minimal account of its properties.

The Johnson homomorphism. Recall the notation $H := H_1(\Sigma_g; \mathbb{Z})$, and observe that there is an embedding

$$H \hookrightarrow \wedge^3 H$$

defined by $x \mapsto x \wedge \omega$, with $\omega = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$ for some symplectic basis $x_1, y_1, \ldots, x_g, y_g$.

Lemma 6.10 (Johnson, [Joh80])**.**

(1) *There is a surjective homomorphism $\tau : \mathcal{I}_g \to \wedge^3 H/H$ known as the* Johnson homomorphism. *It is $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant with respect to the conjugation action on $\mathcal{I}_g$ and the evident action on $\wedge^3 H/H$.*

(2) *Let $c \cup d$ bound a subsurface $\Sigma_{h,2}$. Choose any further subsurface $\Sigma_{h,1} \subset \Sigma_{h,2}$, and let $\{x_1, y_1, \ldots, x_h, y_h\}$ be a symplectic basis for $H_1(\Sigma_{h,1}; \mathbb{Z})$. Then*

$$\tau(T_c T_d^{-1}) = (x_1 \wedge y_1 + \cdots + x_h \wedge y_h) \wedge [c],$$

where c is oriented with $\Sigma_{h,2}$ to the left. In the case $h = 1$, if α, β, γ is a maximal chain on $\Sigma_{1,2}$, then

$$\tau(T_c T_d^{-1}) = [\alpha] \wedge [\beta] \wedge [\gamma].$$

To describe $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g)$, we consider a contraction of $\wedge^3 H/H$. Lemma 6.11 is well known; see, e.g. [Sal19, Lemma 5.6].

Lemma 6.11. *For any s dividing $g-1$, there is an $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant surjection*

$$C_s : \wedge^3 H/H \to H_1(\Sigma_g; \mathbb{Z}/s\mathbb{Z})$$

given by the contraction

$$C(x \wedge y \wedge z) = \langle x, y\rangle z + \langle y, z\rangle x + \langle z, x\rangle y \pmod{s}.$$

Although it was not formulated in this language, Johnson showed that the contraction C vanishes on the group $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g)$.

Lemma 6.12. *Let ϕ be an r–spin structure on a surface Σ_g of genus $g \geq 3$. Set $s = r$ if r is odd, and $s = r/2$ if r is even. Then $C_s \circ \tau = 0$ on $\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g$.*

Proof. We recall (c.f. [Chi72], see also [Joh80, Section 6] and [Sal19, Theorem 5.5]) that the "mod–r Chillingworth invariant" is a homomorphism

$$c_r : \mathcal{I}_g \to 2H_1(\Sigma_g; \mathbb{Z}/r\mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z}/s\mathbb{Z})$$

with the property that $c_r(f) = 0$ for $f \in \mathcal{I}_g$ if and only if f preserves *all* r–spin structures. For r' dividing r, the invariants c_r and $c_{r'}$ are compatible in the sense that $c_{r'} = c_r \pmod{r'}$.

If r is odd, then there is a natural identification of the kernels of c_r and of c_{2r}, for

$$2H_1(\Sigma_g; \mathbb{Z}/2r\mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z}/r\mathbb{Z}) \cong 2H_1(\Sigma_g; \mathbb{Z}/r\mathbb{Z}).$$

Thus it suffices to consider the case of r even.

According to [Joh80, Theorem 3], there is an equality

$$C_{g-1} \circ \tau = c_{2g-2}.$$

This establishes the claim in the case $r = 2g-2$. The general case now follows by reduction mod r. □

We will show that the constraint of Lemma 6.12 in fact *characterizes* the groups $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$ and $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g)$. Lemma 6.13 refines the statement of Lemma 6.4; our goal in the remainder of the subsection is to prove Lemma 6.13 and so accomplish Step 2.

Lemma 6.13. *Set s as in Lemma 6.12. Then there is an equality $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g) = \ker(C_s)$. Consequently, $\tau(\mathrm{Mod}_g[\phi] \cap \mathcal{I}_g) = \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$.*

This will follow by first exhibiting a generating set for $\ker(C_s)$ (Lemma 6.20) and then finding elements of $\mathcal{T}_\phi \cap \mathcal{I}_g$ realizing these elements (Lemma 6.21).

Symplectic linear algebra. To find the generators for $\ker(C_s)$ in $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$, we will make heavy use of the $\mathrm{Sp}(2g, \mathbb{Z})$–equivariance of τ asserted in Lemma 6.10.1. We begin with some results in symplectic linear algebra to this end. We will only need the result of Lemma 6.16 in the proof; the Lemmas 6.14 and 6.15 are preliminary.

Let H be a free $\mathbb{Z}$–module of rank $2g \geq 6$ equipped with a symplectic form $\langle \cdot, \cdot \rangle$, and suppose that q is a nondegenerate quadratic form on $H \otimes (\mathbb{Z}/2Z) \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$ (see Definition 2.10). Given such a q, the *q–vector* of a symplectic basis $\mathcal{B} = (x_1, y_1, \ldots, x_g, y_g)$ for H is the element $\vec{q}(\mathcal{B})$ of $(\mathbb{Z}/2\mathbb{Z})^{2g}$ given by $\vec{q}(\mathcal{B}) = (q([x_1]), \ldots, q([y_g]))$.

Lemma 6.14. *If $\mathcal{B} = (x_1, y_1, \ldots x_g, y_g)$ and $\mathcal{B}' = (x_1', y_1', \ldots, x_g', y_g')$ are symplectic bases with $\vec{q}(\mathcal{B}) = \vec{q}(\mathcal{B}')$, then there is $A \in \mathrm{Sp}(2g, \mathbb{Z})[q]$ such that $A(\mathcal{B}) = \mathcal{B}'$.*

Proof. There is some element $A \in \mathrm{Sp}(2g, \mathbb{Z})$ such that $A(\mathcal{B}) = \mathcal{B}'$. We claim that necessarily $A \in \mathrm{Sp}(2g, \mathbb{Z})[q]$. Let q' be the quadratic form $q' = A \cdot q$. We wish to show that $q' = q$. It suffices to show that $\vec{q'}(\mathcal{B}') = \vec{q}(\mathcal{B}')$. By construction,

$$\vec{q'}(\mathcal{B}') = \vec{q}(A^{-1}\mathcal{B}') = \vec{q}(\mathcal{B}) = \vec{q}(\mathcal{B}');$$

the last equality holding by hypothesis. □

In the statement of Lemma 6.15 below, a *partial symplectic basis* is a collection of vectors $\{v_1, \ldots, v_k\}$ with $\langle v_{2i-1}, v_{2i} \rangle = 1$ for all $2i \leqslant k$ and all other pairings zero. We do not assume that k is even.

Lemma 6.15. *Let q be a quadratic form, $\mathcal{B} = (x_1, y_1, \ldots, x_g, y_g)$ a symplectic basis, and $\vec{q}(\mathcal{B})$ the associated q–vector. Suppose $\{v_1, \ldots, v_k\}$ is a partial symplectic basis, and moreover that $q(v_{2i-1}) = q(x_i)$ and $q(v_{2i}) = q(y_i)$ for all $2i \leqslant k$. Then $\{v_1, \ldots, v_k\}$ admits an extension to a symplectic basis $\mathcal{B}'$ with $\vec{q}(\mathcal{B}) = \vec{q}(\mathcal{B}')$.*

Proof. If k is odd, choose an arbitrary element v_{k+1} satisfying $\langle v_k, v_{k+1} \rangle = 1$ and $q(v_{k+1}) = q(y_{(k+1)/2})$; we proceed with the argument under the assumption that k is even. Let V denote the orthogonal complement to $\{x_1, y_1, \ldots, x_{k/2}, y_{k/2}\}$; this is a symplectic $\mathbb{Z}$–module of rank $2g - k$ equipped with a quadratic form $q|_V$ induced by the restriction of q. Likewise, let W denote the orthogonal complement to $\{v_1, \ldots, v_k\}$; then W is also a symplectic $\mathbb{Z}$–module of rank $2g - k$ equipped with a quadratic form $q|_W$. Since the q–values of $\{x_1, \ldots, y_{k/2}\}$ and $\{v_1, \ldots, v_k\}$ agree and the Arf invariant is additive under symplectic direct sum (Remark 2.13), we conclude that $\mathrm{Arf}(q|_V) = \mathrm{Arf}(q|_W)$. Thus there is a symplectic isomorphism $f : V \to W$ that transports the form $q|_V$ to $q|_W$. The symplectic basis

$$\mathcal{B}' = \{v_1, \ldots, v_k, f(x_{k/2+1}), f(y_{k/2+1}), \ldots, f(x_g), f(y_g)\}$$

satisfies $\vec{q}(\mathcal{B}) = \vec{q}(\mathcal{B}')$ by construction. □

Lemma 6.16.

(1) *Let v_1, v_2, v_3 and v_1', v_2', v_3' be partial symplectic bases for H. If $q(v_i) = q(v_i')$ for $i = 1, 2, 3$, then there is some element $A \in \mathrm{Sp}(2g, \mathbb{Z})[q]$ such that $A(v_1 \wedge v_2 \wedge v_3) = v_1' \wedge v_2' \wedge v_3'$.*

(2) *Let v_1, v_2, v_3 and v_1', v_2', v_3' be triples such that $\langle v_i, v_j \rangle = \langle v_i', v_j' \rangle = 0$ for all pairs of indices $1 \leqslant i < j \leqslant 3$. If $q(v_i) = q(v_i')$ for $i = 1, 2, 3$, then there is some element $A \in \mathrm{Sp}(2g, \mathbb{Z})[q]$ such that $A(v_1 \wedge v_2 \wedge v_3) = v_1' \wedge v_2' \wedge v_3'$.*

Proof. For (1), we extend $\{v_1, v_2, v_3\}$ and $\{v_1', v_2', v_3'\}$ to symplectic bases $\mathcal{B}, \mathcal{B}'$. By Lemma 6.15, we can furthermore assume that $\vec{q}(\mathcal{B}) = \vec{q}(\mathcal{B}')$. The required element $A \in \mathrm{Sp}(2g, \mathbb{Z})[q]$ is now obtained by an appeal to Lemma 6.14.

The proof of (2) is very similar: $\{v_1, v_2, v_3\}$ and $\{v_1', v_2', v_3'\}$ can again be extended to symplectic bases $\mathcal{B}, \mathcal{B}'$ with equal q–vectors and the result follows by Lemma 6.14. □

Some topological computations. Along with symplectic linear algebra, we will also need to see that $\mathcal{T}_\phi$ contains an ample supply of certain specific mapping classes.

Lemma 6.17. *Let ϕ be an r–spin structure on a surface Σ_g of genus $g \geq 3$ (if $g = 3$, assume $\mathrm{Arf}(\phi) = 1$). Let $b \subset \Sigma_g$ be a nonseparating simple closed curve satisfying $\phi(b) = -1$. Then $T_b^r \in \mathcal{T}_\phi$.*

Proof. This will require a patchwork of arguments depending on the specific values of r and g. For $g \geq 5$ and $r < g - 1$, this was established in [Sal19, Lemma 5.2]. We will treat the remaining cases as follows: (1) for $g \geq 3$ and $r = 2g - 2$, (2) for $g \geq 4$ and $r = g - 1$, (3) the remaining sporadic cases appearing for $g \leqslant 4$.

(1): Lemmas 5.10 and 5.13 furnish a specific b with $\phi(b) = -1$ and $T_b^{2g-2} \in \mathcal{T}_\phi$. By the change–of–coordinates principle (specifically Lemma 6.8), given *any* nonseparating b' satisfying $\phi(b') = -1$, one can find an element $f \in \mathrm{Mod}_g[\phi]$ such that $f(b') = b$; consequently the elements T_b^{2g-2} and $T_{b'}^{2g-2}$ are conjugate elements of $\mathrm{Mod}_g[\phi]$. To conclude the argument, we observe that $\mathcal{T}_\phi$ is a *normal* subgroup of $\mathrm{Mod}_g[\phi]$, so that $T_{b'}^{2g-2} \in \mathcal{T}_\phi$ as desired.

(2): Assume now $g \geq 4$ and $r = g - 1$. Let b be an arbitrary nonseparating curve satisfying $\phi(b) = -1$. Our first task is to find a certain configuration of admissible curves well–adapted to b; the D relation (Lemma 5.9) then allows us to exhibit $T_b^r \in \mathcal{T}_\phi$. The configuration we construct is depicted in Figure 15.

By the change–of–coordinates principle (Lemma 6.6 or 6.7), there exists a chain of admissible curves $a_2, \ldots, a_{2g-3}$ disjoint from b. Let a_1' be any curve satisfying $i(a_1', b) = i(a_1', a_2) = 1$ and $i(a_1', a_j) = 0$ for $j \geq 2$. For any $k \in \mathbb{Z}$, the curves $T_b^k(a_1')$ have these same intersection properties. Since $\phi(b) = -1$, twist linearity (Definition 2.1.1) implies that we can choose $a_1 = T_b^k(a_1')$ for suitable k such that a_1 is admissible. Finally, let a_0 be a curve so that $a_0 \cup a_2 \cup b$ forms a pair of pants to the left of b, and such that $i(a_0, a_3) = 1$ and $i(a_0, a_j) = 0$ for all other j. By homological coherence (Lemma 2.3), a_0 is also admissible. Finally, let d be chosen so that $b \cup a_2 \cup a_4 \cup \cdots \cup a_{2g-8} \cup d$ bounds a subsurface to the left of b of genus 2 and $g - 2$ boundary components, such that $i(d, a_{2g-7}) = 1$ and $i(d, a_j) = 0$ for other j. Since $r = g - 1$, homological coherence (Lemma 2.3) implies that d is admissible.

Consider the $\mathscr{D}_{2g-3}$ configuration determined by the curves $a_0, a_2, \ldots, a_{2g-3}$. By construction, one boundary component is b, and the other is the curve c shown in Figure 15. Applying the D relation to this configuration shows that

$$T_b^{2g-5} T_c \in \mathcal{T}_\phi. \tag{4}$$

Consider next the $\mathscr{D}_{2g-6}$ configuration determined by the curves $a_0, a_2, \ldots a_{2g-7}, d$. This configuration has boundary components b, c, and the *separating* curve c'. Applying the D relation shows

$$T_b^{g-4} T_c T_{c'} \in \mathcal{T}_\phi; \tag{5}$$

since c' is separating, we invoke Lemma 6.4 to conclude that also $T_b^{g-4} T_c \in \mathcal{T}_\phi$. Combining (4) and (5) shows that $T_b^{g-1} \in \mathcal{T}_\phi$.

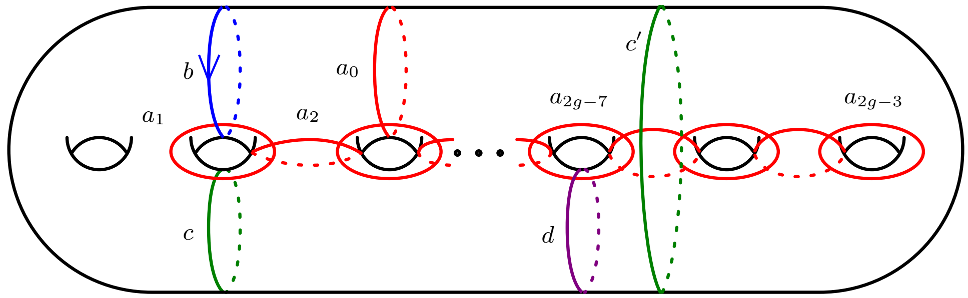


FIGURE 15. The configuration used in the proof of Lemma 6.17.

(3) The remaining cases are $(g, r) = (3, 2)$ and $(4, 2)$. Let b be a nonseparating curve satisfying $\phi(b) = -1$, and choose an admissible curve a_1 disjoint from b. Let a_2 be chosen so that $a_1 \cup a_2 \cup b$ forms a pair of pants; by homological coherence (Lemma 2.3), a_2 is also admissible. By the change–of–coordinates principle (Lemmas 6.6 and 6.7) , it is easy to find an admissible curve a_0 with the following intersection properties:

$$i(a_0, b) = 0, \qquad i(a_0, a_1) = i(a_0, a_2) = 1.$$

Finally, choose a_3 so that the following conditions are satisfied: $i(a_3, a_0) = 1$ and a_3 is disjoint from all other curves under consideration, and $a_1 \cup a_3$ bounds a subsurface of genus 1 containing b. By homological coherence, a_3 is admissible, and by construction, (a_0, a_1, a_2, a_3) forms a $\mathscr{D}_4$ configuration. In the notation of Figure 7, the boundary component Δ_0 is separating, and the curves Δ_1 and Δ_1' are both isotopic to b. By the D relation (Lemma 5.9),

$$T_{\Delta_0} T_b^2 \in \mathcal{T}_\phi.$$

By Lemma 6.4, since Δ_0 is separating, it follows that $T_b^2 \in \mathcal{T}_\phi$ as required. □

The notion of a "curve–arc sum" is a convenient language for building new curves on surfaces from old ones. This construction will be used in Lemma 6.19 below.

Definition 6.18 (Curve–arc sum)**.** Let a and b be disjoint oriented simple closed curves, and let ε be an embedded arc connecting the left side of a to that of b so that ε is otherwise disjoint from $a \cup b$. A regular neighborhood of $a \cup \varepsilon \cup b$ is then a three–holed sphere; two of the boundary components are isotopic to a and b. The *curve–arc sum* $a +_\varepsilon b$ is the simple closed curve in the isotopy class of the third boundary component. See Figure 16.

Lemma 6.19. *Let (x_i, y_i) and (x_j, y_j) be distinct pairs of symplectic basis vectors, and let $z \in H$ be a primitive vector orthogonal to $\langle x_i, y_i, x_j, y_j\rangle$; if r is even, suppose $q(z) = 1$. Then there is an element $f \in \mathcal{T}_\phi \cap \mathcal{I}_g$ satisfying*

$$\tau(f) = z \wedge (x_i \wedge y_i - x_j \wedge y_j).$$

Proof. This follows the argument for (G2) given in [Sal19, proof of Lemma 5.8]. The change–of–coordinates principle (in the guise of Lemma 6.9) implies that there exists an admissible curve c

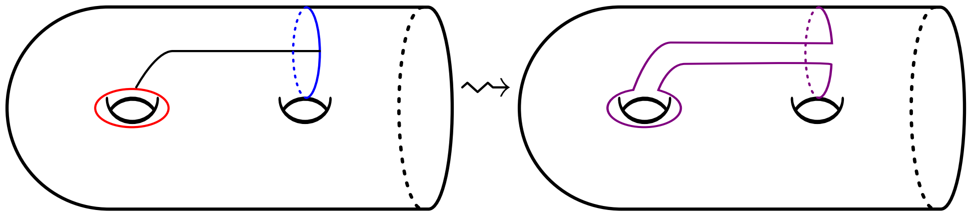


FIGURE 16. The curve–arc sum operation.

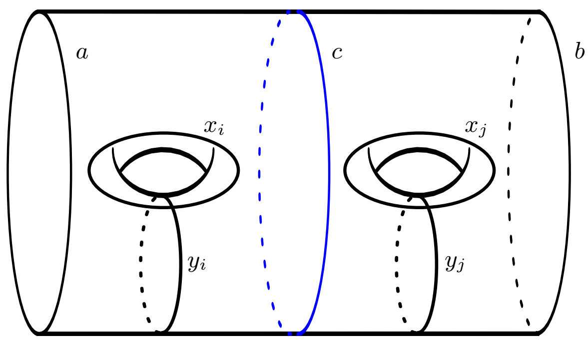


FIGURE 17. The configuration of curves used to exhibit (G2).

such that $[c] = z$ (in the case of r even, this uses the assumption that $q(z) = 1$). By the classical change–of–coordinates principle, there exist curves a, b with the following properties: (1) $a \cup b$ bounds a subsurface S of genus 2, (2) $c \subset S$, (3) $[a] = [b] = [c]$, and c separates S into two subsurfaces S_1, S_2 each of genus 1, (4) x_i, y_i determine a symplectic basis for S_1 and x_j, y_j determine a symplectic basis for S_2. Such a configuration is shown in Figure 17. By homological coherence (Lemma 2.3), $\phi(a) = \phi(b) = -2$ when a, b are oriented with S to the left. By Lemma 6.10.2,

$$\tau(T_a T_c^{-1}) = z \wedge x_i \wedge y_i$$

and

$$\tau(T_b T_c^{-1}) = -z \wedge x_j \wedge y_j.$$

Therefore, it is necessary to show $T_a T_b T_c^{-2} \in \mathcal{T}_\phi$. By hypothesis, $T_c \in \mathcal{T}_\phi$, so it remains to show $T_a T_b \in \mathcal{T}_\phi$ as well.

We claim that there exists a maximal chain $a_1, \ldots, a_5$ of admissible curves on S; modulo this, the claim follows by an application of the chain relation. Choose an arbitrary subsurface $S' \subset S$ homeomorphic to Σ_2^1, and let $\mathcal{B} = \{c_1, c_2, c_3, c_4\}$ be a geometric symplectic basis for S'; by Lemma 6.6.1 or 6.7.1, such a basis can be chosen with c_1, c_2, c_3 admissible, and c_4 either admissible or else satisfying $\phi(c_4) = -1$.

If $\phi(c_4) = 0$, consider the curve–arc sum

$$c_4' = c_4 +_\varepsilon a,$$

where ε is disjoint from c_1, c_2, c_3. By homological coherence (Lemma 2.3), $\phi(c_4') = -1$, and $\{c_1, c_2, c_3, c_4'\}$ forms a geometric symplectic basis. Thus we may assume that S' is chosen with c_1, c_2, c_3 admissible and $\phi(c_4) = -1$. Under this assumption, we can set $a_1 = c_1, a_2 = c_2, a_4 = c_3$, and $a_3 = c_4 +_\varepsilon a_2$ with ε an arc connecting the left side of c_4 to a_1 and otherwise disjoint from the other curves under consideration. By homological coherence, a_3 is admissible as well. Now let a_5 be any curve extending $a_1, \ldots, a_4$ to a maximal chain on S. Since $\phi(a) = -2$ when oriented with S to the left, homological coherence implies that a_5 is admissible, and we have constructed the required maximal chain. ☐

Concluding Lemma 6.13. We can now show that $\mathcal{T}_\phi \cap \mathcal{I}_g$ surjects onto $\ker(C_s)$. Note that establishing Lemma 6.21 will complete the proof of Lemma 6.13, which in turn completes the final Step 3 of the proof of Proposition 6.1.

Lemma 6.20. *For any s dividing $g-1$, the subspace $\ker(C_s) \leqslant \wedge^3 H/H$ has a generating set consisting of the following classes of elements; in each case $z \in \{x_1, y_1, \ldots, x_g, y_g\}$ with further specifications listed below.*

(G1) $s(z \wedge x_i \wedge y_i)$ *for* $z \neq x_i, y_i$

(G2) $z \wedge (x_i \wedge y_i - x_j \wedge y_j)$ *for* $z \neq x_i, y_i, x_j, y_j$

(G3) $z_i \wedge z_j \wedge z_k$ *for* $\{i, j, k\} \subset \{1, \ldots, g\}$ *distinct.*

Proof. See [Sal19, proof of Lemma 5.8]. ☐

Lemma 6.21. *Let ϕ be an r–spin structure on a surface Σ_g of genus $g \geq 3$ (if $g = 3$, assume $\mathrm{Arf}(\phi) = 1$). For each generator f of the form (G1) – (G3) as presented in Lemma 6.20, the group $\mathcal{T}_\phi \cap \mathcal{I}_g$ contains an element γ satisfying $\tau(\gamma) = f$.*

Proof. To avoid having to formulate two nearly identical arguments, one for each parity of r, we treat only the case of r even. The presence of a residual mod–2 spin structure makes this case strictly harder than that for r odd.

Let q denote the quadratic form associated to ϕ; recall that if c is a simple closed curve, then $q([c]) = \phi([c]) + 1 \pmod 2$. We fix a symplectic basis $\mathcal{B} = \{x_1, y_1, \ldots, x_g, y_g\}$ such that $q(x_i) = 1$ for $1 \leqslant i \leqslant g$ and $q(y_i) = 1$ for $1 \leqslant i \leqslant g-1$; the value of $q(y_g)$ is then determined by $\mathrm{Arf}(q)$. Throughout, we will use the following principle: we will perform a topological computation to obtain some tensor in $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g) \leqslant \wedge^3 H/H$. We will then combine the $\mathrm{Sp}(2g, \mathbb{Z})$–equivariance of Lemma 6.10.1 and the surjectivity result $\Psi(\mathcal{T}_\phi) = \mathrm{Sp}(2g, \mathbb{Z})[q]$ of Lemma 6.3 to see that this single computation provides a large class of further elements of $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$.

Generators of type (G1) are of the form $s(z \wedge x_i \wedge x_j)$; here $s = r/2$. To obtain such elements in $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$, we begin by using the change–of–coordinates principle (Lemma 6.7.4) to choose a 3–chain of admissible curves a_0, a_1, a_2 representing respectively the homology classes $x_1, y_1, (x_1 + x_2 + x_3)$. Let b, b' denote the boundary components of this chain. By the chain relation, $T_b T_{b'} \in \mathcal{T}_\phi$ and so

$$(T_b T_{b'})^s \in \mathcal{T}_\phi$$

as well. By homological coherence, $\phi(b) = -1$, and so by Lemma 6.17, also

$$T_b^r \in \mathcal{T}_\phi$$

Combining these two shows that

$$(T_b T_{b'}^{-1})^s \in \mathcal{T}_\phi.$$

By Lemma 6.10.2, it follows that

$$s\ (x_1 \wedge y_1 \wedge (x_2 + x_3)) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g).$$

This argument can be repeated with curves a_0', a_1', a_2' representing respectively the homology classes $x_1, y_1, (x_1 + y_2 + x_3)$, showing that also

$$s\ (x_1 \wedge y_1 \wedge (y_2 + x_3)) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g).$$

Subtracting,

$$s\ (x_1 \wedge y_1 \wedge (x_2 - y_2)) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g).$$

As $q(x_1) = q(y_1) = q(x_2 - y_2) = 1$, Lemma 6.16.1 shows that $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$ contains all generators of type (1) of the form $s\ (z \wedge x_i \wedge y_i)$ for $i \leqslant g-1$, except for $z = y_g$ in the case $q(y_g) = 0$. In this latter case, an application of Lemma 6.16.1 to $s\ (x_1 \wedge y_1 \wedge (x_2 + x_3))$ shows that $s\ (y_g \wedge x_i \wedge y_i) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$ for $i \leqslant g-1$ regardless.

It remains to show $s\ (z \wedge x_g \wedge y_g) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$. If $q(y_g) = 1$ then the above results are already sufficient. Otherwise, by above,

$$s\ (x_1 \wedge x_g \wedge (y_{g-1} + y_g)) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g).$$

It thus suffices to show $s\ (x_1 \wedge x_g \wedge y_{g-1}) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$. By Lemma 6.16.2, it is in turn sufficient to show $s\ (x_1 \wedge x_2 \wedge x_3) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$. By the computations above,

$$s\ (x_1 \wedge (y_1 + x_2 + x_3) \wedge x_3) \qquad \text{and} \qquad s\ (x_1 \wedge y_1 \wedge x_3)$$

are both elements of $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$; taking the difference, the result follows.

Now we consider generators of type (G2); recall these are of the form $z \wedge (x_i \wedge y_i - x_j \wedge y_j)$. Applying Lemma 6.19, we find

$$z \wedge (x_i \wedge y_i - x_j \wedge y_j) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$$

for x_i, y_i, x_j, y_j arbitrary and for all $z \in \{x_1, \ldots, y_g\}$ satisfying $q(z) = 1$. This encompasses all elements $x_1, \ldots, x_{g-1}, y_{g-1}, x_g$, and possibly y_g as well. In the case where $q(y_g) = 0$, we have $q(y_{g-1}) = q(y_{g-1} + y_g) = 1$. Applying Lemma 6.19 with $z = y_{g-1}$ and $z = y_{g-1} + y_g$ in turn and subtracting, we obtain all elements of the form

$$y_g \wedge (x_i \wedge y_i - x_j \wedge y_j) \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$$

as well, completing this portion of the argument.

Finally, we consider generators of type (G3), of the form $z_i \wedge z_j \wedge z_k$ for distinct indices i, j, k. By Lemma 6.9, there exists a curve d with $[d] = x_2$ and $\phi(d) = -2$. Choose some curve e_1 disjoint from d such that $d \cup e_1$ bounds a subsurface S_1 of genus 1 to the left of d, and such that S_1 contains a pair of curves in the homology classes x_1, y_1. By homological coherence (Lemma 2.3), e_1 is admissible, and by Lemma 6.10.2,

$$\tau(T_d T_{e_1}^{-1}) = x_1 \wedge y_1 \wedge x_2.$$

Similarly, we can find a curve e_2 disjoint from d such that $d \cup e_2$ bounds a subsurface S_2 of genus 1 to the left of d, and such that S_2 contains a pair of curves in the homology classes $x_1, y_1 - x_3$. Again by homological coherence, e_2 is admissible, and by Lemma 6.10.2,

$$\tau(T_d T_{e_2}^{-1}) = x_1 \wedge (y_1 - x_3) \wedge x_2.$$

Combining these computations, since e_1, e_2 are admissible,

$$\tau(T_{e_2} T_{e_1}^{-1}) = x_1 \wedge x_3 \wedge x_2 \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g).$$

Applying Lemma 6.16.2, it follows that if z_i, z_j, z_k are pairwise–orthogonal primitive vectors with $q(z_i) = q(z_j) = q(z_k) = 1$, then $z_i \wedge z_j \wedge z_k \in \tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$. This includes all generators of the form (G3) except when $z_i = y_g$ and $q(y_g) = 0$. In this case, both $q(y_{g-1}) = q(y_{g-1} + y_g) = 1$, and we conclude the argument as we did for (G2) by finding $z_i \wedge z_j \wedge y_{g-1}$ and $z_i \wedge z_j \wedge (y_{g-1} + y_g)$ in $\tau(\mathcal{T}_\phi \cap \mathcal{I}_g)$ and subtracting. □

This completes the proof of Proposition 6.1, and hence the proof of Theorem B1 and 2.

6.5. **The case of general r.** In this section, we prove Theorem B3. We first demonstrate how the change–of–coordinates principle and twist–linearity can be used, given two curves, to produce a third whose winding number is the greatest common divisor of the other two.

Lemma 6.22. *Let ϕ be a $(2g-2)$–spin structure on a surface Σ_g of genus at least 3 and suppose that $\phi(a_1) = k_1$ and $\phi(a_2) = k_2$. Set*

$$\Gamma = \langle \mathrm{Mod}_g[\phi], T_{a_1}, T_{b_2} \rangle.$$

Then Γ contains T_c for some nonseparating curve c with $\phi(c) = \gcd(k_1, k_2)$.

Proof. Set $r = \gcd(k_1, k_2)$; then there exist some $x, y \in \mathbb{Z}$ such that

$$xk_1 + yk_2 = r.$$

Without loss of generality, we may suppose that $x, y \neq 0$ (else T_{a_1} or T_{a_2} has the desired property.).

By the change–of–coordinates principle (Lemma 6.7), there exists some curve b_1 with $i(b_1, a_2) = 1$ and $\phi(b_1) = k_1$. By Lemma 6.8, there is some element $f \in \mathrm{Mod}_g[\phi]$ such that $f(a_1) = b_1$; then $T_{b_1} = f T_{a_1} f^{-1}$ and so $T_{b_1} \in \Gamma$. Now by twist linearity (Definition 2.1(1)), we have that

$$\phi\left(T_{b_1}^x(a_2)\right) = \phi(a_2) + x\phi(b_1) = xk_1 + k_2.$$

Again by Lemma 6.7 , there is a curve b_2 which only intersects $T_{b_1}^x(a_2)$ once and has $\phi(b_2) = k_2$. Applying Lemma 6.8 as above, we similarly see that $T_{b_2} \in \Gamma$. Therefore

$$\phi\left(T_{b_2}^{(y-1)}\left(T_{b_1}^x(a_2)\right)\right) = \phi\left(T_{b_1}^x(a_2)\right) + (y-1)\phi(b_2) = xk_1 + yk_2 = r.$$

Setting $c = T_{b_2}^{(y-1)} T_{b_1}^x(a_2)$ completes the proof (since c is in the Γ orbit of a_2, the twist T_c is conjugate to T_{a_2} by an element of Γ). □

Proof of Theorem B3. Let ϕ, $\widetilde{\phi}$, and $\{c_i\}$ be as in the statement of the theorem, and set

$$\Gamma = \langle \mathrm{Mod}_g[\widetilde{\phi}], \{T_{c_i}\}\rangle.$$

Our task is to show that $\Gamma = \mathrm{Mod}_g[\phi]$. By Proposition 6.1, it is enough to show that $\Gamma = \mathcal{T}_\phi$, i.e. that $T_c \in \Gamma$ for any curve c with $\phi(c) = 0$. Since $\widetilde{\phi}$ is a lift of ϕ, we see that this is equivalent to showing that $T_c \in \Gamma$ for every c with $\widetilde{\phi}(c) \equiv 0 \pmod r$.

We claim that it is enough to exhibit the Dehn twist in a single curve c with $\widetilde{\phi}(c) = r$. Indeed, by the transitivity of $\mathrm{Mod}_g[\widetilde{\phi}]$ on the set of (non–separating) curves with given $\widetilde{\phi}$–winding number (Lemma 6.8), this implies that every curve with winding number r is in Γ. Now given any c with $\widetilde{\phi}(c) \equiv 0 \pmod r$, Lemma 6.7.4 guarantees that there exists some curve d with $\widetilde{\phi}(d) = r$ which intersects c exactly once, and by twist–linearity (Definition 2.1), we have that

$$\widetilde{\phi}\left(T_d^{-(\widetilde{\phi}(c)/r)}(c)\right) = \widetilde{\phi}(c) - (\widetilde{\phi}(c)/r) \cdot r = 0.$$

Therefore T_c may be conjugated to a $\widetilde{\phi}$–admissible twist by an element of Γ and hence $T_c \in \Gamma$.

To exhibit such a twist, one needs only to iteratively apply Lemma 6.22 to the collection of curves $\{c_i\}$ to recover a curve with $\widetilde{\phi}$–winding number $r = \gcd\left(\widetilde{\phi}(c_i)\right)$. □

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