Mapping class groups, vector fields, and holomorphic differentials

Aaron Calderon

Yale University

(joint work w/ Nick Salter)

“Geometric” subgroups of Mod(S) often have surprising finiteness properties

- Stabilizer of curves/ subsurfaces
- Point-pushing subgroups
- Hyperelliptic MCGs (Birman–Hilden)
- Handlebody groups (Suzuki, Wajnryb)
- Torelli group (Johnson)
- Johnson filtration (Ershov–He, Church–Ershov–Putman)
- Monodromy of families of curves over varieties
- Today: Framed MCGs

Framed surfaces and MCGs

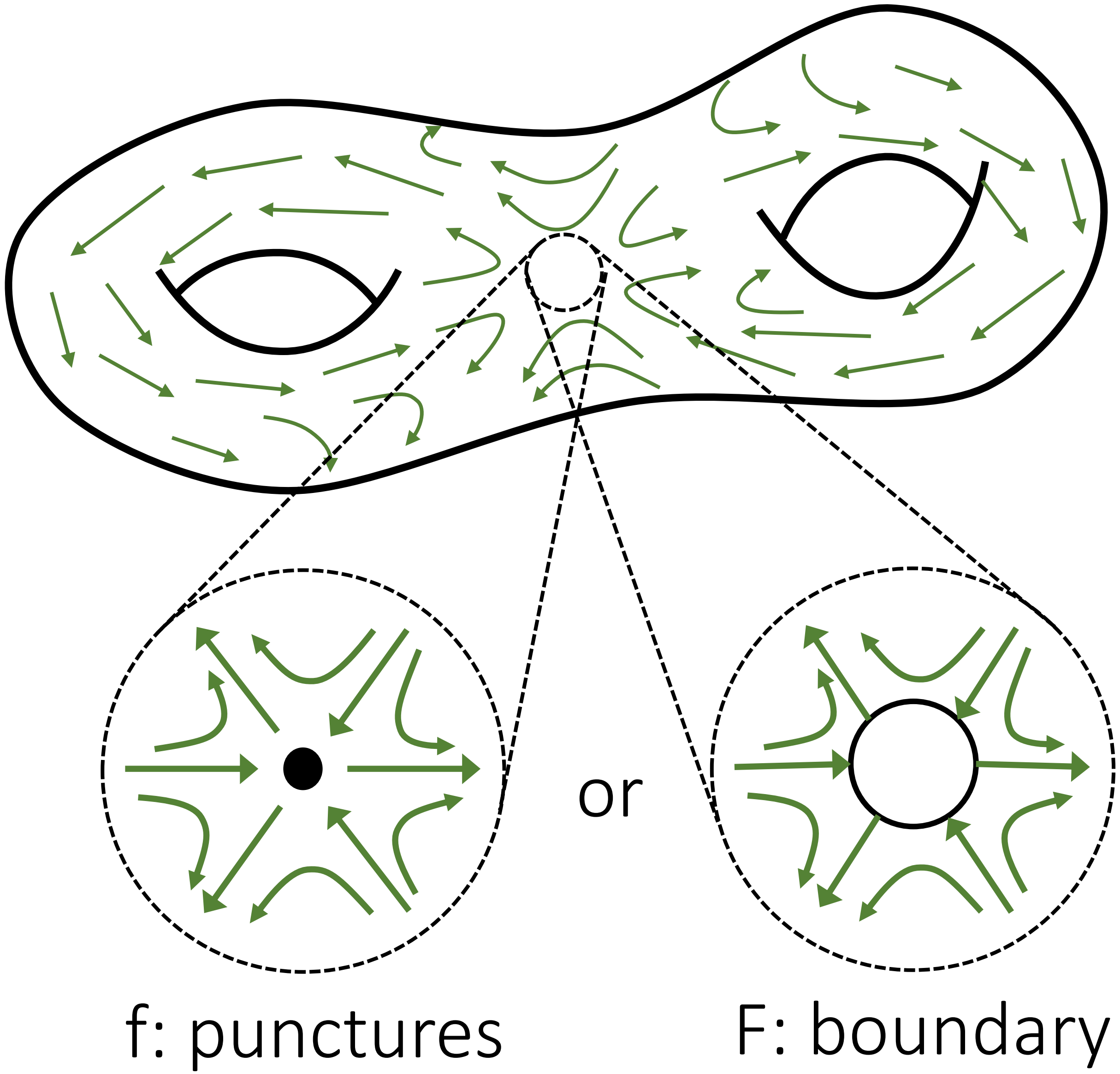
Framed surfaces

Framing ⇝ nonzero vector field

Signature = $(k_1, \ldots, k_n)$
$k_i = -$ (index of i^{th} boundary)

Poincaré–Hopf: $\Sigma\, k_i = 2g-2$

winding number functions:
wn(f): {scc} $\longrightarrow \mathbb{Z}$
wn(F): {scc} ⊔ {arcs} $\longrightarrow \mathbb{Z}$



Framed surfaces

<u>Twist-linearity:</u>

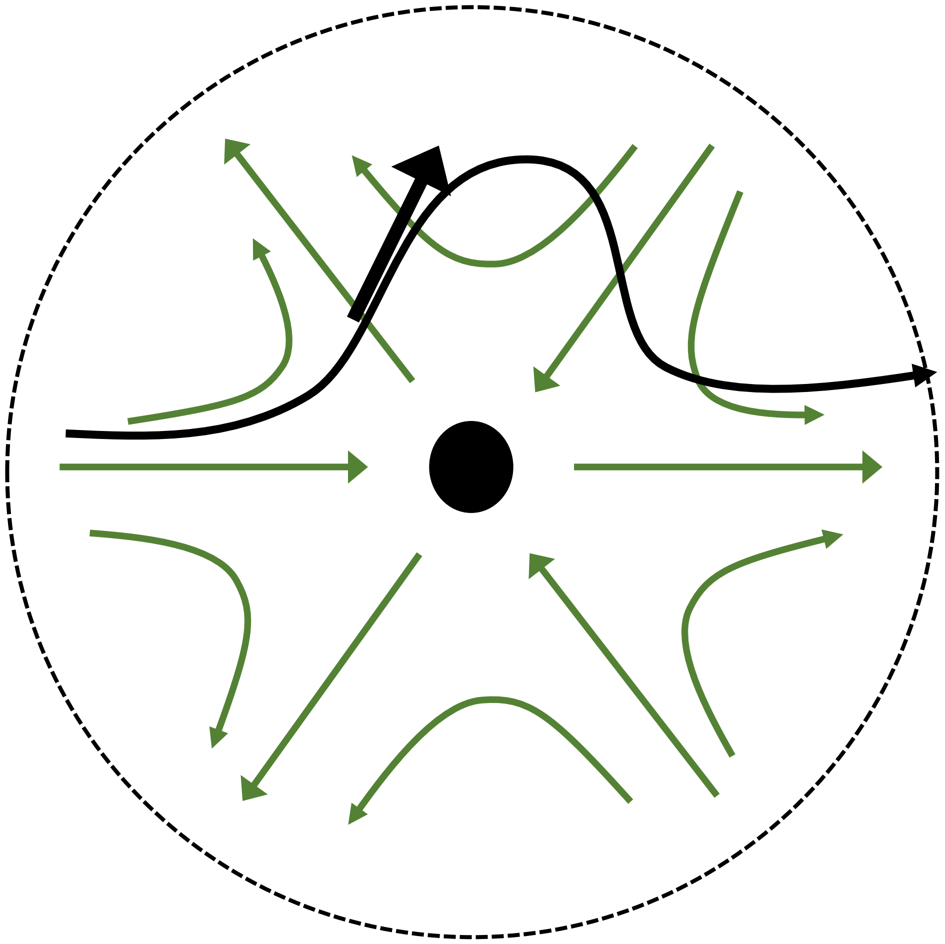
γ an arc or curve, c a curve.

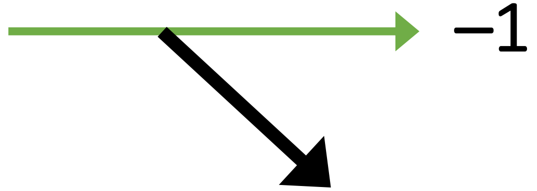
$\mathrm{wn}(T_c\,\gamma) = \mathrm{wn}(\gamma) + \langle \gamma, c \rangle\, \mathrm{wn}(c)$

winding number functions:

$\mathrm{wn}(f)$: $\{\text{scc}\} \longrightarrow \mathbb{Z}$

$\mathrm{wn}(F)$: $\{\text{scc}\} \sqcup \{\text{arcs}\} \longrightarrow \mathbb{Z}$





Framed mapping class groups

<u>Twist-linearity:</u>
γ an arc or curve, c a curve.
$wn(T_c \gamma) = wn(\gamma) + \langle \gamma, c \rangle\, wn(c)$

Framed MCGs:
$Mod_g^n[f]$ and $Mod_{g,n}[F]$
(punctures) (boundary)

wn(c) = 0 "admissible"
$\rightsquigarrow T_c \in Mod_g^n[f]$ and $Mod_{g,n}[F]$

- stabilizes f/F up to isotopy
- preserves all winding #s

c separating
$\rightsquigarrow T_c \in Mod_g^n[f]$ but *not* $Mod_{g,n}[F]$

Infinite index, not normal

Theorem: [C.–Salter, '20]

Let $\underline{k}$ be a partition of $2g-2$ with $g \geq 5$.

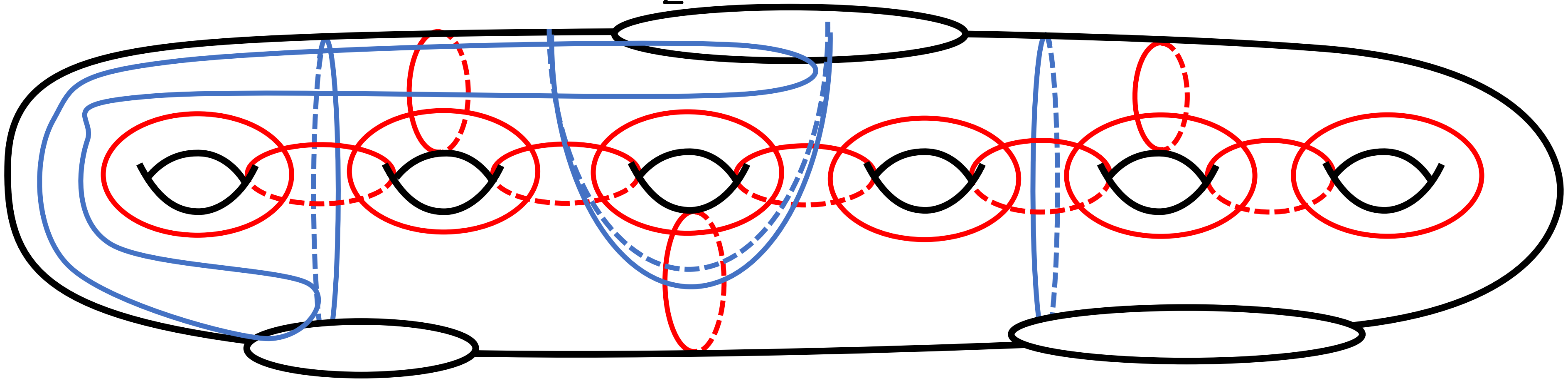
Every framed mapping class group of signature $\underline{k}$ is generated by an* explicit finite set of Dehn twists.

$\mathrm{Mod}_{g,n}[F]$: admissible twists
$\mathrm{PMod}_g^n[f]$: admissible + separating twists

*actually, we give a *very* general criterion for when a set of Dehn twists generates

Generating sets

e.g. signature (4,3,3) ⇝ genus 6



$\mathrm{Mod}_{g,n}[F]$: admissible twists

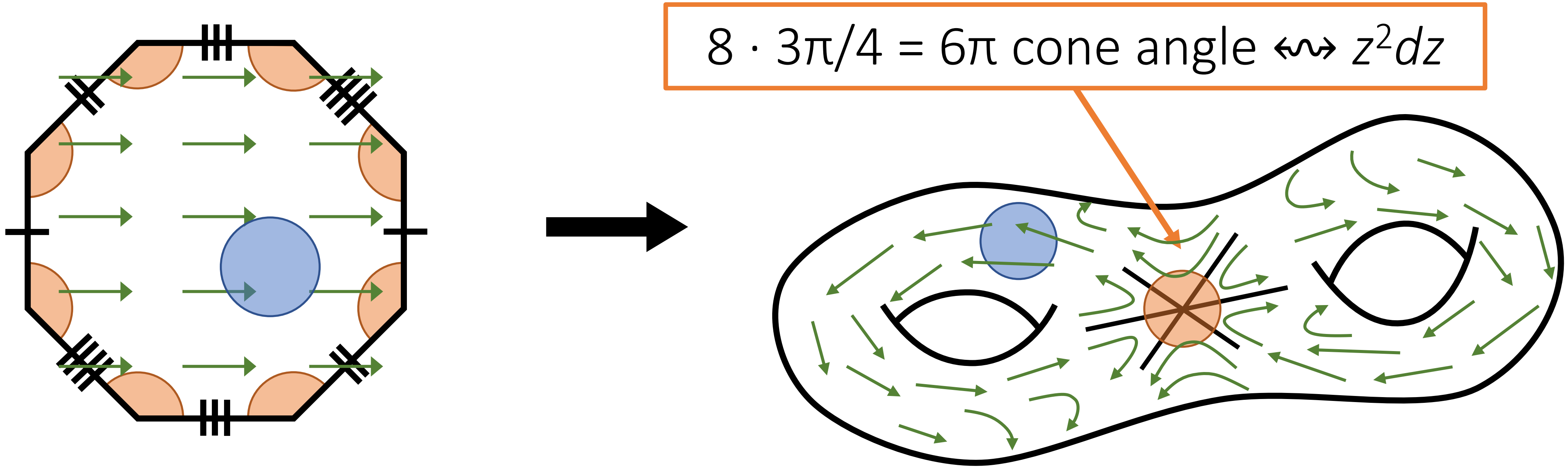
$\mathrm{PMod}_g^n[f]$: admissible + separating twists

Application: plane curve singularities [Portilla-Cuadrado & Salter]

(see Nick's talk on Sunday!)

Today's application:
strata of Abelian differentials

Abelian differentials

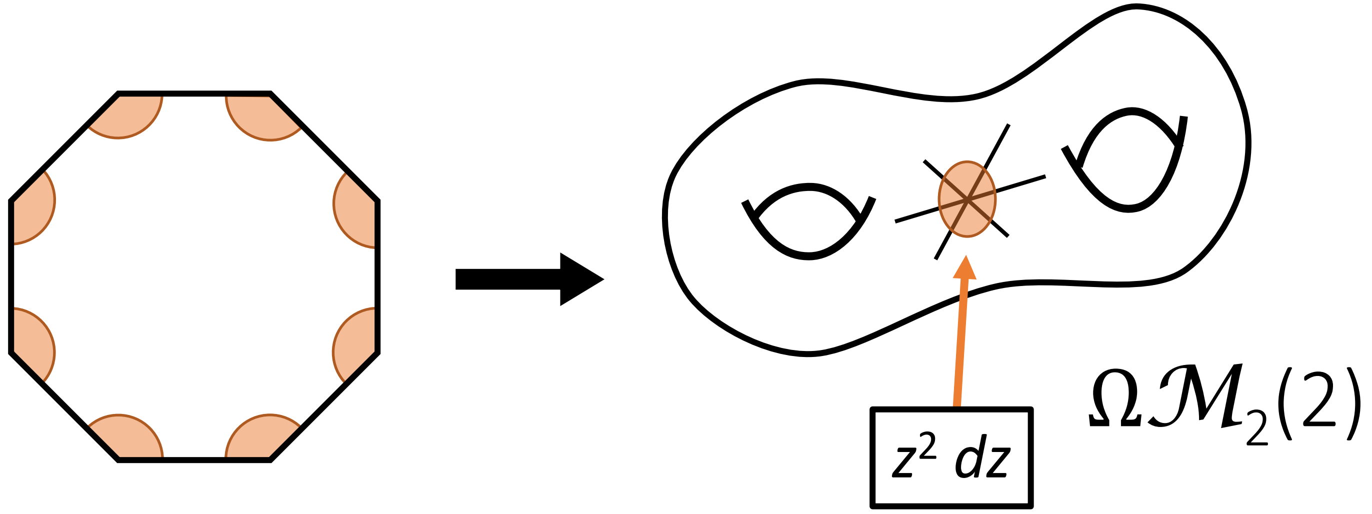


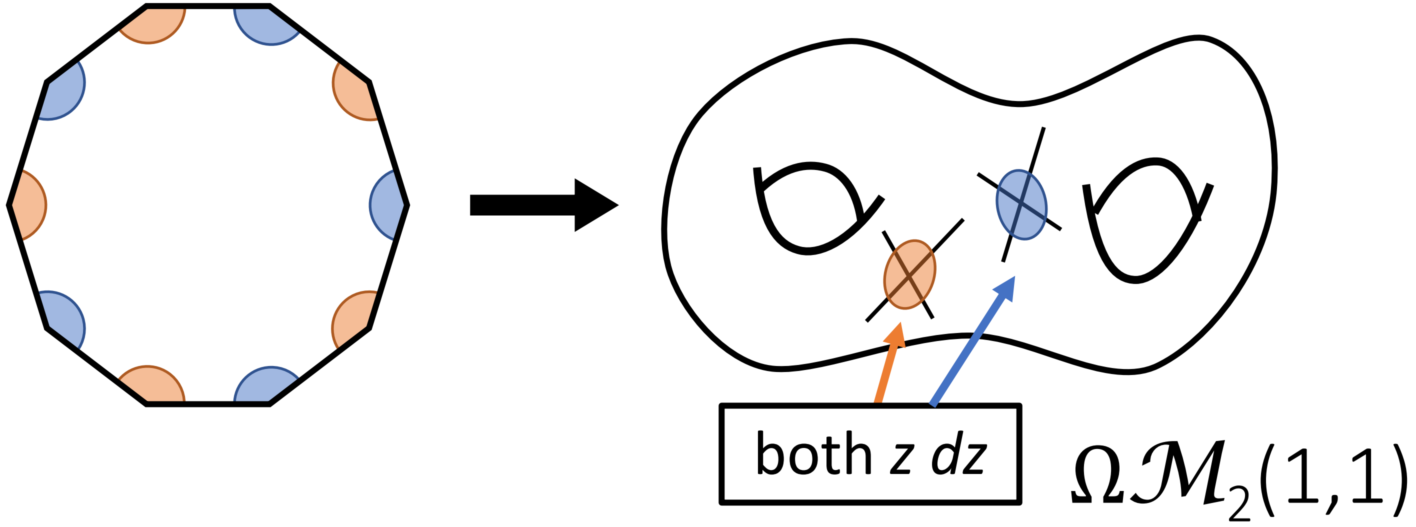
Abelian differential ω = holomorphic 1-form on Riemann surface

- Flat cone metric: cone angle ⇜⇝ zero order
- Horizontal vector field 1/ω ⇜⇝ framing of S \ Zeros(ω)

Strata

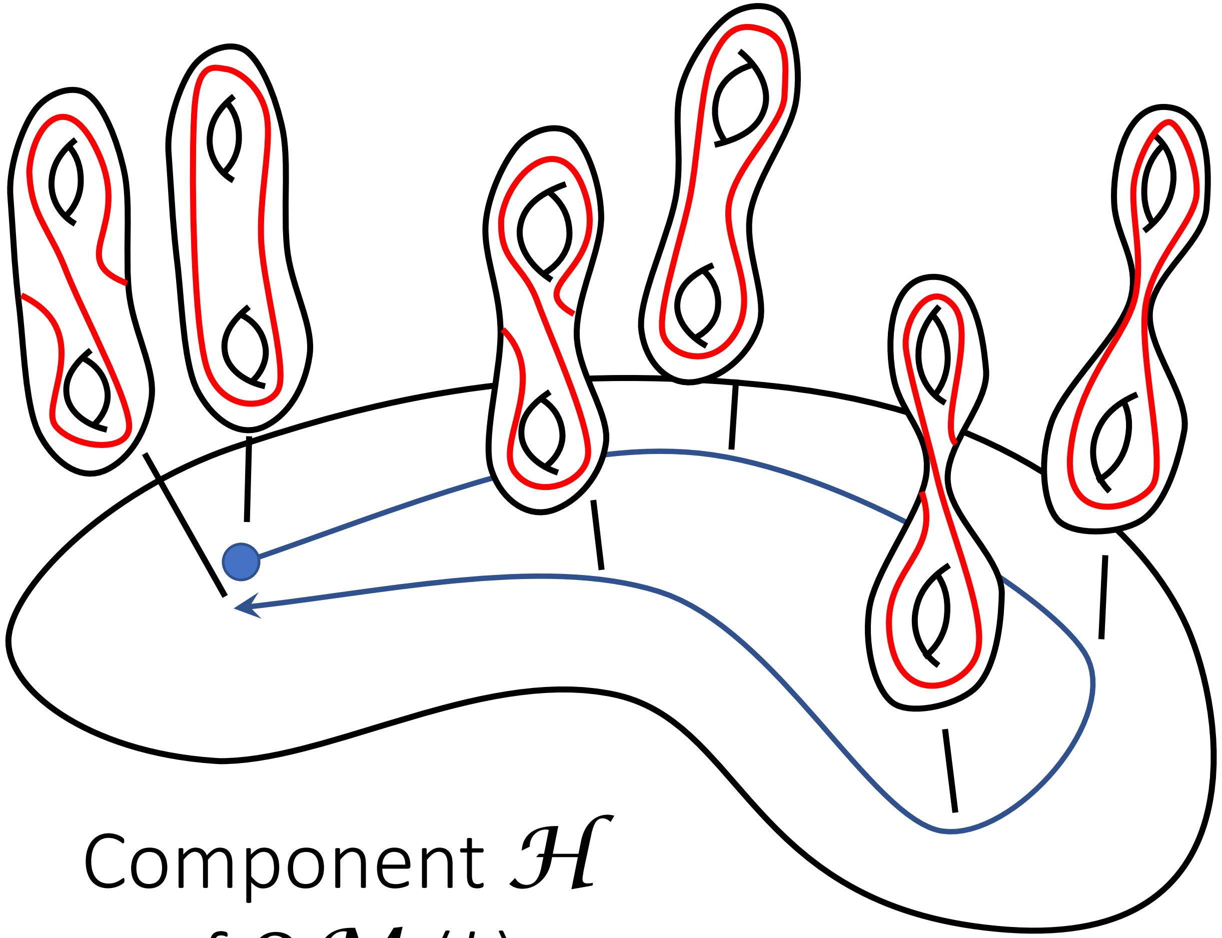
$\Omega\mathcal{M}_g(k_1, \ldots, k_n)$ =
moduli space of ω's with
zeros of orders $k_1, \ldots, k_n$
⇝ Framing 1/ω has
signature $(k_1, \ldots, k_n)$





<u>Conjecture:</u>
[Kontsevich–Zorich]
Strata are K(π,1)'s for
"some sort of MCG"

Monodromy of strata



Component $\mathcal{H}$ of $\Omega\mathcal{M}_g(\underline{k})$

Loops in $\Omega\mathcal{M}_g(\underline{k})$ induce diffeomorphisms of S preserving Zeros(ω)

Monodromy maps:

$\pi_1(\mathcal{H}) \to \mathrm{Mod}_g$

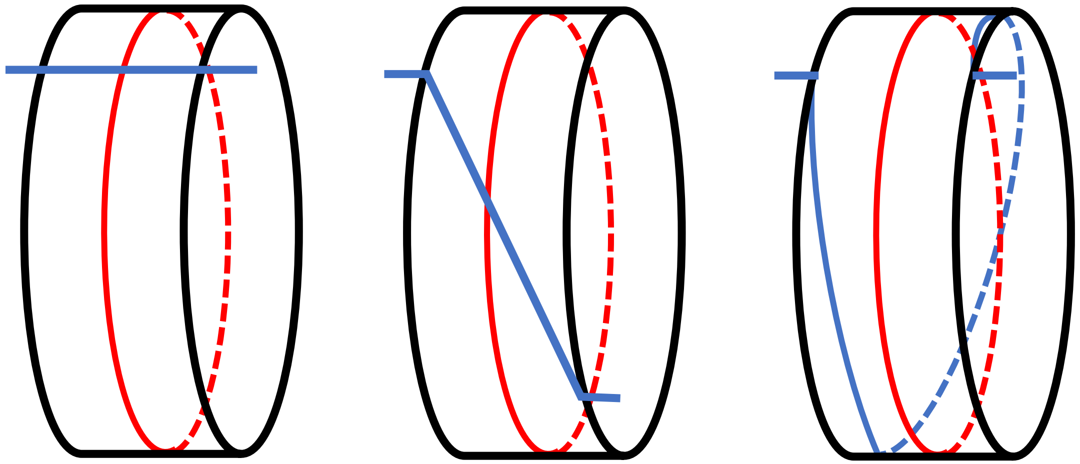
$\pi_1(\mathcal{H}) \to \mathrm{Mod}_g^n$

VI (f.i.)

$\pi_1(\mathcal{H}') \dashrightarrow \mathrm{Mod}_{g,n}$

"real oriented blow-up"

Monodromy of strata



<u>Explicit example:</u>

cylinder shears ⟶ Dehn twists

Cylinders are admissible ⟹ in $\mathrm{Mod}_{g,n}[1/\omega]$

Loops in $\Omega\mathcal{M}_g(\underline{k})$ induce diffeomorphisms of S preserving Zeros(ω)

Monodromy maps:

$\pi_1(\mathcal{H}) \to \mathrm{Mod}_g$

$\pi_1(\mathcal{H}) \to \mathrm{Mod}_g^n$

VI (f.i.)

$\pi_1(\mathcal{H}') \dashrightarrow \mathrm{Mod}_{g,n}$

"real oriented blow-up"

Theorem: [C.–Salter, '20]

Let $\underline{k}$ be a partition of $2g-2$ with $g \geq 5$.

Let $\mathcal{H}$ be a nonhyperelliptic component of $\Omega\mathcal{M}_g(\underline{k})$.

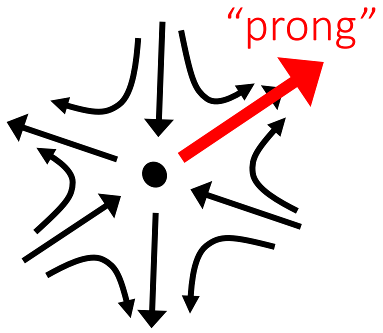
Then the image of the monodromy homomorphism

$$\pi_1(\mathcal{H}, \omega) \longrightarrow \mathrm{Mod}_g^n$$

is exactly the framed mapping class group $\mathrm{Mod}_g^n[1/\omega]$.

(the hyperelliptic case is both rare and classical)

Track:	Parallel transport:	Stabilizes:
zeros	surface with punctures	framing (∼isotopy)
prong	surface with boundary	framing (∼ relative isotopy)
all prongs	"pronged" surface	framing (∼isotopy preserving prongs)
only surface [C.–Salter, '19]	closed surface	framing (mod $r = \gcd(\underline{k})$) = "r-spin structure"
only homology	$H_1(S, \mathrm{Zeros}(\omega))$	total mod 2 winding numbers



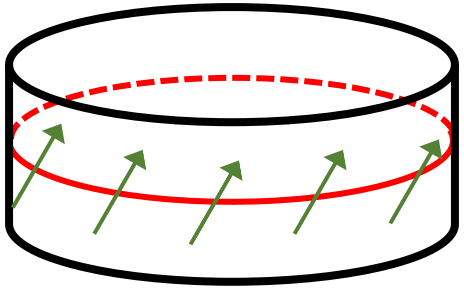
Tracking different data
⇝ different monodromy maps

"The monodromy always stabilizes some sort of framing"

Corollaries

> "The only obstructions to being realized geometrically are topological"

Cylinders are all admissible.



"Saddle" = nonsingular geodesic connecting cone points.

Conversely, *every admissible curve can be made a cylinder.*

Every arc between distinct zeros can be made a saddle.

<u>i.e.</u> if c is admissible, there is a path* γ in $\mathcal{H}$ so that the parallel transport of c along γ is a cylinder.

<u>i.e.</u> for such a, there is a path** γ in $\mathcal{H}$ so that the parallel transport of a along γ is a saddle.

* actually, a *loop*

** can't always use a loop!

Sketch of monodromy theorem*

<u>Theorem:</u> [C.–Salter]
The image of monodromy is the framed MCG.

<u>Step 1:</u> Monodromy ≤ framed MCG

As differential ω deforms, so does 1/ω

⟹ Winding numbers are continuous, hence constant

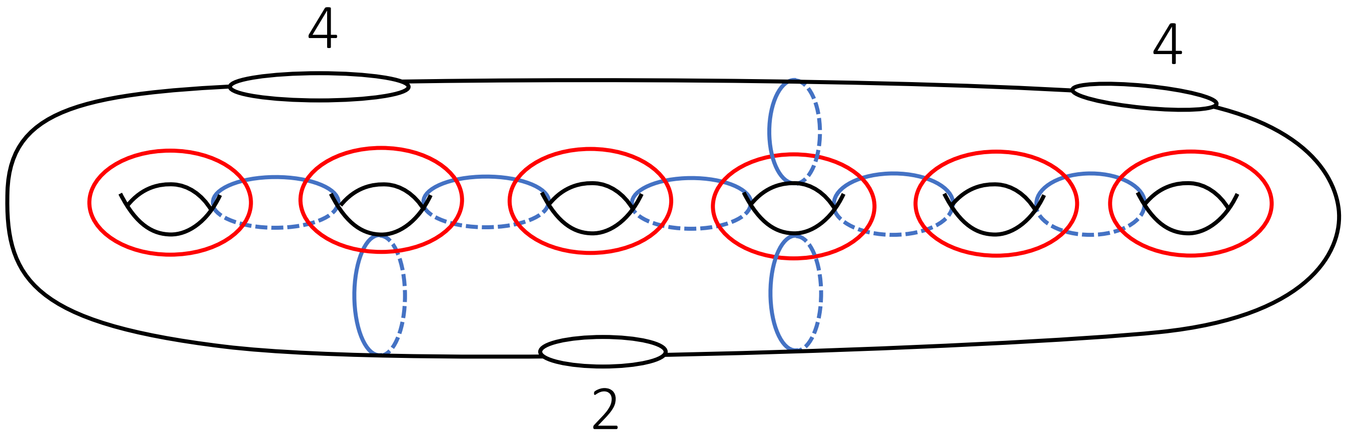
<u>Step 2:</u> Monodromy ≥ framed MCG

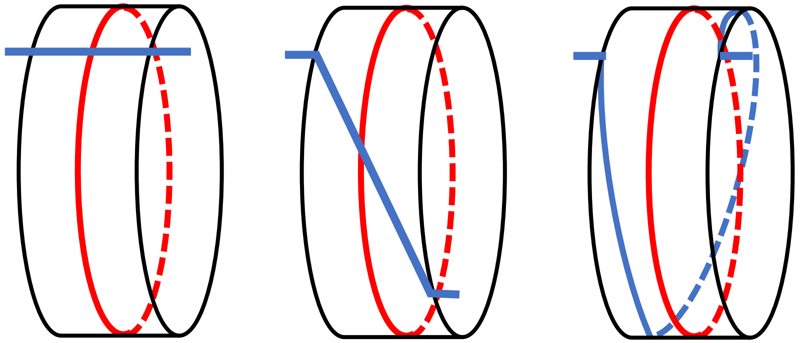
Build loops with prescribed monodromy (flat geometry)

Apply generating set theorem*

Building loops with prescribed monodromy

To build Dehn twists, use cylinder shears!





[Thurston–Veech]

i. Find a pair of multicurves adapted to the stratum
ii. Dual complex
    = square–ulation
iii. Make each square flat
iv. Curves become cylinders

Sketch of generating set theorem

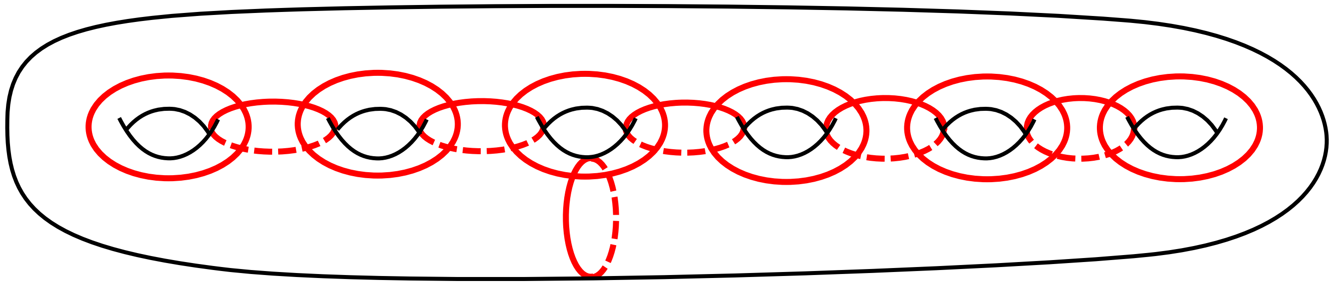
idea: induct by merging zeros

<u>Base case:</u>

"Framed BES:" $1 \longrightarrow [\pi_1, \pi_1] \longrightarrow \mathrm{PMod}_g[F] \longrightarrow \mathrm{Mod}_g[\varphi] \longrightarrow 1$

⇝ reduce to closed surface, use [C.–Salter '19]

wn only defined mod $2g-2$
φ = "$(2g-2)$-spin structure"



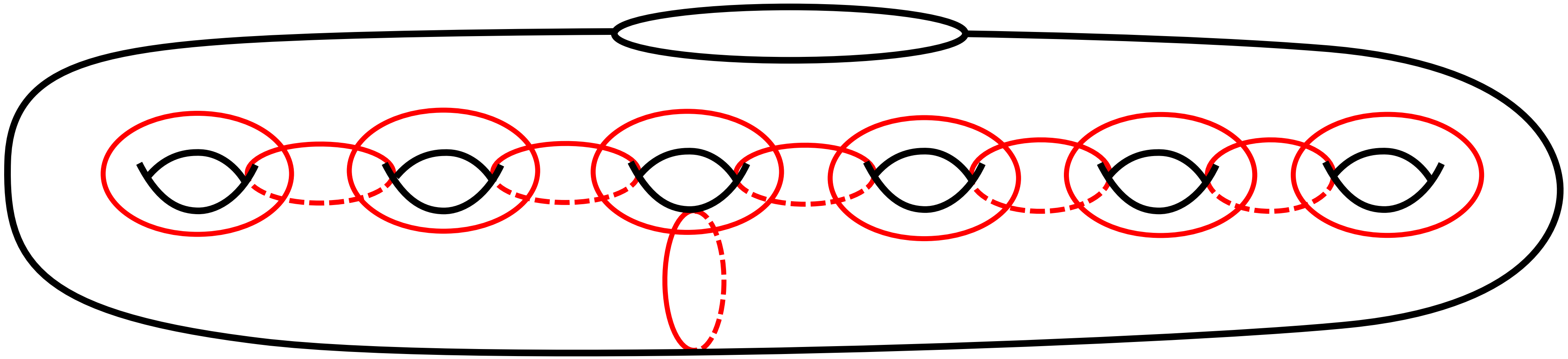
<u>Inductive step:</u>

Merge zeros along arc

⇝ need to understand connectedness of an arc complex

The inductive step

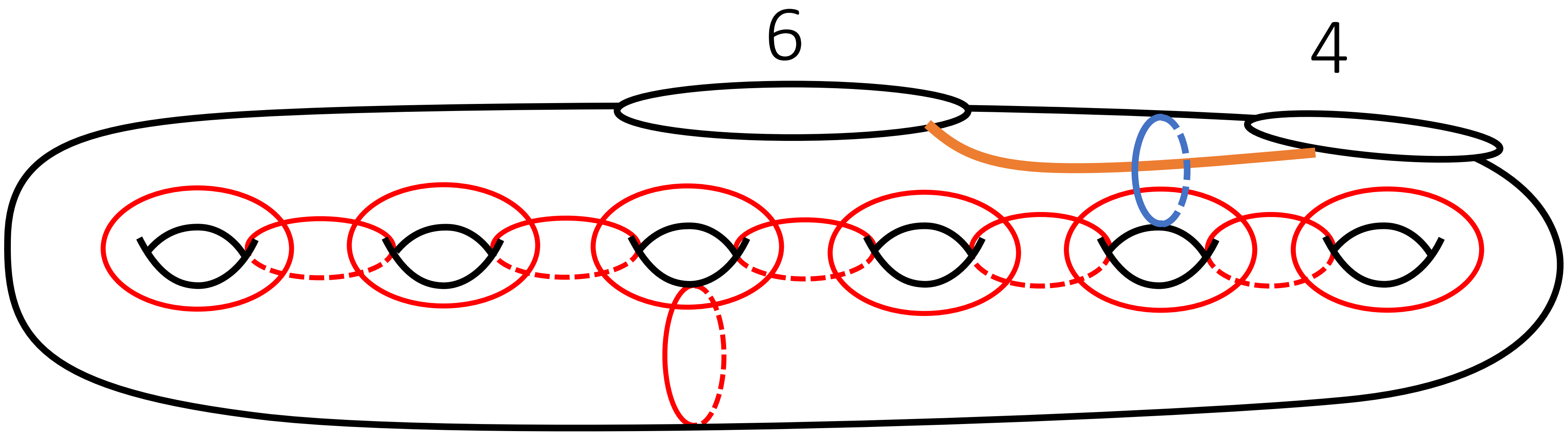
<u>*e.g.*</u> signature (4,3,3) ⇜ (4,6) ⇜ (10)



<u>Base case:</u> these curves generate $\mathrm{Mod}_{g,1}[10]$

The inductive step

e.g. signature (4,3,3) ⇜ (4,6) ⇜ (10)

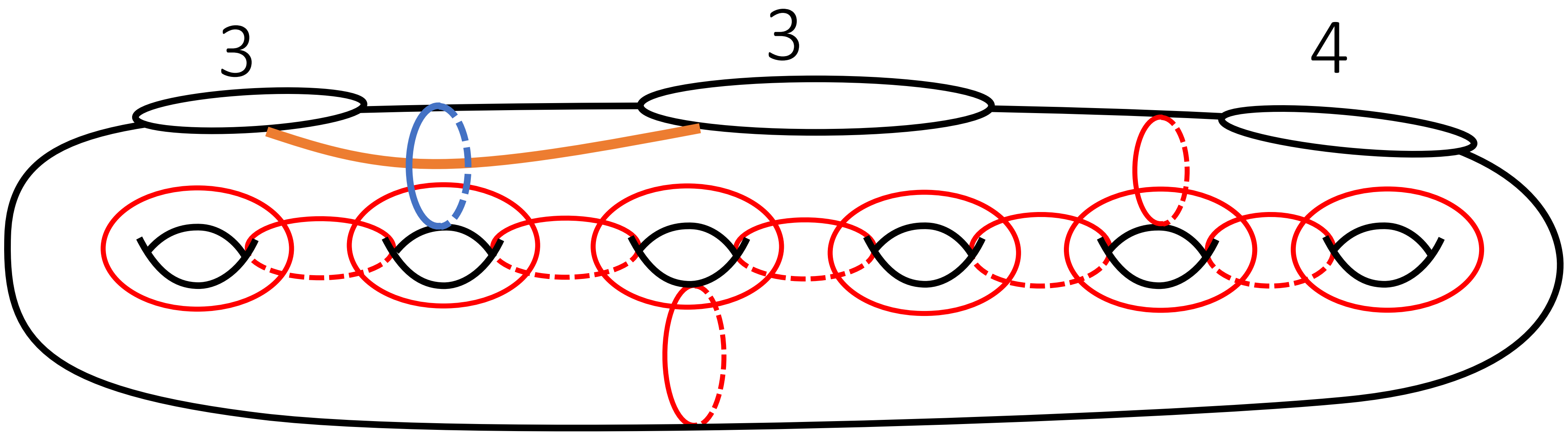


Stab(arc)=$\mathrm{Mod}_{g,1}[10]$=⟨admissible twists⟩
New admissible moves over edge in complex

⟹ $\mathrm{Mod}_{g,2}[4,6]$ = ⟨these twists⟩

The inductive step

e.g. signature (4,3,3) ⇜ (4,6) ⇜ (10)

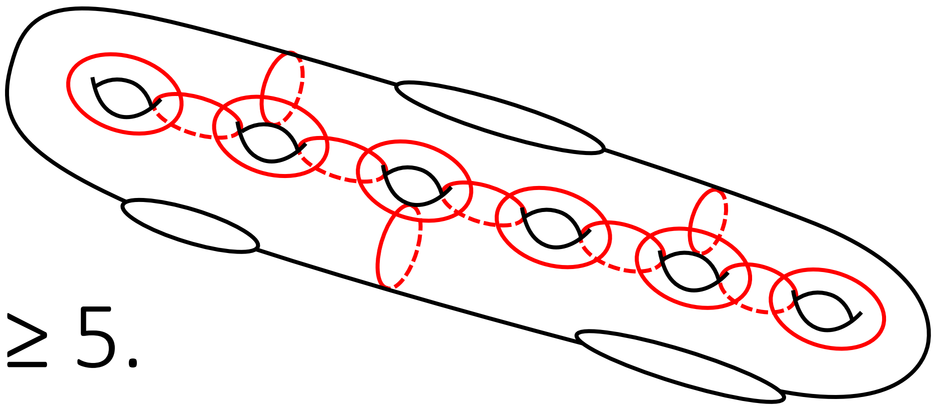


Stab(arc)=$\text{Mod}_{g,2}[4,6]$=⟨admissible twists⟩

New admissible moves over edge in complex

⟹

$\text{Mod}_{g,3}[4,3,3]$ = ⟨these twists⟩



<u>Theorem:</u> [C.–Salter, '20]

Let $\underline{k}$ be a partition of $2g-2$ with $g \geq 5$.

Let $\mathcal{H}$ be a nonhyperelliptic component of $\Omega\mathcal{M}_g(\underline{k})$.

Then the image of the monodromy homomorphism

$$\pi_1(\mathcal{H},\omega) \longrightarrow \mathrm{Mod}_g^n$$

is exactly the framed mapping class group $\mathrm{Mod}_g^n[1/\omega]$.